# HOMEWORK 11 — DUE DEC 15TH

MATH 25

There are two sections: A and B. Please return each part to the proper CA.

## A. Problems graded by Benjamin

In the following 4 problems, $K$ is a field which contains $\mathbf{Q}$ and $M_n(K)$ is the set of all $n \times n$ matrices.

A.1. **Matrices of trace 0 I.** Let $M \in M_n(K)$ be a matrix whose trace is 0.

(1) Prove that there exists an invertible matrix $P$ such that the first column of $P^{-1}MP$ is
$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
(2) Prove that there exists an invertible matrix $Q$ such that all the diagonal entries of $Q^{-1}MQ$ are 0.

A.2. **Matrices of trace 0 II.** Let
$$D = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}$$
and let $\varphi : M_n(K) \to M_n(K)$ be the map defined by $\varphi(M) = MD - DM$.

(1) Prove that $\ker\varphi$ is the set of diagonal matrices. What is the rank of $\varphi$?
(2) Prove that $\operatorname{im}\varphi$ is a subspace of the space of matrices whose diagonal entries are 0.
(3) Prove that $\operatorname{im}\varphi$ is the space of matrices whose diagonal entries are 0.

A.3. **Matrices of trace 0 III (Shoda's theorem).** Using the previous two problems, prove that a matrix $M$ satisfies $\mathrm{Tr}(M) = 0$ if and only if there exist $A, B \in M_n(K)$ such that $M = AB - BA$.

A.4. **Matrices of trace 0 IV.** Prove that the subspace of $M_n(K)$ generated by all nilpotent matrices is the space of matrices whose trace is 0.

A.5. **Skolem-Noether's theorem.** Let $\varphi : M_n(K) \to M_n(K)$ be a bijective map such that $\varphi(A + \lambda B) = \varphi(A) + \lambda\varphi(B)$, $\varphi(\mathrm{Id}) = \mathrm{Id}$ and $\varphi(AB) = \varphi(A)\varphi(B)$. Such a map is called an automorphism of $M_n(K)$. The goal of the problem is to find all automorphisms of $M_n(K)$.

(1) Check that if $P$ is an invertible matrix, then the map $\varphi_P$ defined by $\varphi_P(M) = PMP^{-1}$ is an automorphism of $M_n(K)$.
(2) Let $E_{i,j}$ be the matrix all of whose entries are 0 except the $(i, j)$-th which is 1. Compute $E_{i,j}E_{k,\ell}$ (the answer depends on whether $j = k$ or not).
(3) Now let $\varphi$ be an unspecified automorphism and let $U_{i,j} = \varphi(E_{i,j})$. Prove that $\ker(U_{1,1} - \mathrm{Id})$ is non-trivial.
(4) Let $f_1 \in \ker(U_{1,1} - \mathrm{Id})$ be a non-trivial element and for $k \geq 2$, set $f_k = U_{k,1}(f_1)$. Prove that $\{f_1, \cdots, f_n\}$ is a basis of $K^n$.
(5) Let $P$ be the matrix of $\{f_i\}$ in the usual basis of $K^n$. Prove that $\varphi$ and $\varphi_P$ coincide on the $E_{k,1}$'s.
(6) prove by recurrence on $n$ that if $\varphi$ is an automorphism of $M_n(K)$, then there exists an invertible matrix $Q$ such that $\varphi(M) = QMQ^{-1}$.

## B. Problems graded by Inna

B.1. **Symmetries.** Let $f \in \mathrm{End}(V)$ be a map such that $f^2 = \mathrm{Id}$.

(1) Prove that $f$ is diagonalizable, with eigenvalues $\pm 1$. Let $V_\pm$ be the eigenspace corresponding to $\pm 1$.
(2) Why is such a map called a symmetry (your answer should involve $V_\pm$)?

B.2. Prove that if $f \in \mathrm{End}(V)$, then $V = \ker(f) \oplus \mathrm{im}(f)$ if and only if 0 is a simple root of $\Pi_f$.

B.3. **Projections.** Let $f \in \mathrm{End}(V)$ be a map such that $f^2 = f$.

(1) Prove that $\ker(f) \oplus \mathrm{im}(f) = V$.
(2) Prove that $f$ is diagonalizable. What are its eigenvalues, and what are the corresponding eigenspaces?
(3) Why is such a map called a projection?

B.4. Let $\mathbf{C}_n[X]$ be the space of polynomials with complex coefficients and degree $\leq n$. Prove that the map $f : \mathbf{C}_n[X] \to \mathbf{C}_n[X]$ defined by $f(P) = (X^2 - 1)P''$ is diagonalizable.

B.5. Let $A$ and $B$ be two matrices and let

$$M = \begin{pmatrix} A & \\ & B \end{pmatrix}.$$

Find $\Pi_M$ in terms of $\Pi_A$ and $\Pi_B$.

B.6. Prove that if $f : E \to E$ is a diagonalizable map and if $V \subset E$ is a subspace stable by $f$, then the restriction of $f$ to $V$ is diagonalizable. Prove an analogous statement for "triangulable".

B.7. Let $M, N \in M_n(K)$ be such that $MN - NM = \lambda M$ where $\lambda \neq 0$.

(1) Prove that $M^k N - NM^k = k\lambda M^k$ for all $k \geq 0$.
(2) Prove that $M$ is nilpotent.