HOMEWORK 9 — DUE NOV 21ST

MATH 25

There are two sections: A and B. Please return each part to the proper CA.
In all these problems, if R is a ring, $R^* = R \setminus \{0\}$.

A. Problems graded by Benjamin

A.1. Let (E, d) be a metric space, and let $f : E \to \mathbf{C}$ be a function. Write $f(x) = a(x) + ib(x)$ where a, b are two functions from E to $\mathbf{R}$.

(1) prove that f is continuous if and only if a and b are continuous.
(2) prove that if f, g are two continuous functions from (E, d) to $\mathbf{C}$ then $f \pm g$ and $f \cdot g$ are also continuous.

A.2. If $m \in \mathbf{Z}_{\geq 1}$ let $\phi(m) = \text{card}\{1 \leq x \leq m,\ \gcd(x, m) = 1\}$. If G is a group and if $g \in G$, the order of g is the smallest $m \geq 1$ such that $g^m = e$. Don't confuse this with the order of G which is $\text{card}\, G$.

(1) prove that the order of g is the order of the subgroup it generates in G (assume that G is finite).
(2) prove that if G is any finite group and if $g \in G$ then the order of g divides the order of G.
(3) prove that if $n \geq 1$, then $\sum_{d|n} \phi(d) = n$ (here $\sum_{d|n}$ means the sum over all divisors d of n).
(4) show that if G is a cyclic group of order n and d divides n, then there are $\phi(d)$ elements in g of order d.
(5) in (5)-(7) let K be a field and let G be a finite subgroup of $(K^*, \times)$ of order n. Prove that there are at most d elements $g \in G$ such that $g^d = 1$.
(6) prove that there exists an element $g \in G$ which is of order n.
(7) prove that G is cyclic.
(8) let G be the set of invertible elements in $\mathbf{Z}/24\mathbf{Z}$. Prove that G is not a cyclic group.

A.3. Let S^1 be the set of complex numbers z such that $|z| = 1$ and let C_n be the set of complex numbers such that $z^n = 1$.

(1) prove that $(S^1, \times)$ and $(C_n, \times)$ are compact subgroups of $(\mathbf{C}^*, \times)$.
(2) are there any other ones?

A.4. If $P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbf{C}[X]$, let R be the largest of the $|z_i|$ where $z_1, \cdots, z_d$ are the roots of P in $\mathbf{C}$. Prove the following inequalities:

(1) if $r > 0$ satisfies $r^d \geq \sum_{i=0}^{d-1} |a_i| r^i$, then $R \leq r$.
(2) $R \leq \max(1, \sum_{i=1}^{d-1} |a_i|)$ (Montel)
(3) $R \leq 1 + \max_{1 \leq k \leq d} |a_k|$ (Cauchy)
(4) $R \leq |1 - a_{d-1}| + |a_{d-1} - a_{d-2}| + \cdots + |a_1 - a_0| + |a_0|$ (Montel)
(5) if $a_i \in \mathbf{R}$ for all i and $1 \geq a_{d-1} \geq a_{d-2} \geq \cdots \geq a_0 \geq 0$ then $R \leq 1$ (Kakeya)

B. Problems graded by Inna

B.1. Find a formula for $\sum_{k=0}^{n} \cos(kx)$ and for $\sum_{k=0}^{n} \sin(kx)$. Hint: consider $1 + e^{ix} + \cdots + e^{inx}$.

B.2. Prove that if $P \in \mathbf{C}[X]$ is a non-constant polynomial, then the function $z \mapsto |P(z)|$ has no local maximum in $\mathbf{C}$ (a local maximum is a point $z_0 \in \mathbf{C}$ such that there exists $\delta > 0$ with the property: if $|z - z_0| < \delta$, then $|P(z)| \leq |P(z_0)|$).

B.3. Let $P(X) = a_0 + a_1X + \cdots + a_dX^d \in \mathbf{R}[X]$ be a polynomial of degree d. We say that P is split in $\mathbf{R}$ if one can write $P(X) = a_d(X - \alpha_1) \cdots (X - \alpha_d)$ where the α_i's are (not necessarily distinct) elements of $\mathbf{R}$. For example $X^2 - 2X + 1$ is split but not $X^2 + 1$.

(1) prove that if P is split then P' is split.
(2) more generally, prove that if $\lambda \in \mathbf{R}$, and if P is split, then $P' + \lambda P$ is split (hint: consider the function $x \mapsto P(x)e^{\lambda x}$).
(3) prove that if a polynomial $b_0 + b_1X + \cdots + b_dX^d$ and $P(X)$ are both split then so is the polynomial $b_0P(X) + b_1P'(X) + \cdots + b_dP^{(d)}(X)$.

B.4. If $z_1, \cdots, z_d$ are complex numbers, the convex hull of the z_i's is the set of complex numbers of the form $z = \lambda_1 z_1 + \cdots + \lambda_d z_d$ where $\lambda_i \in \mathbf{R}_{\geq 0}$ and $\lambda_1 + \cdots + \lambda_d = 1$.

(1) for $n = 2, 3, 4, 5$ describe the convex hull of the n solutions of the equation $z^n = 1$ (draw a picture!).
(2) Let $P \in \mathbf{C}[X]$ be a polynomial of degree d, whose roots are $z_1, \cdots, z_d$. Prove that

$$\frac{P'(X)}{P(X)} = \sum_{i=1}^{d} \frac{1}{X - z_i}$$

(3) use this to show that the roots of $P'(X)$ lie in the convex hull of the roots of $P(X)$.