HOMEWORK 8 — DUE NOV 14TH

MATH 25

There are two sections: A and B. Please return each part to the proper CA.

A. Problems graded by Benjamin

A.1. In each of the following exercises, you are given a set G and a composition law $(x, y) \mapsto x \otimes y$. Determine whether $(G, \otimes)$ is a group and if it is, find the unit element, give a formula for the inverse of an element, and determine whether G is abelian.

(1) $G = \mathbf{R}$ and $x \otimes y = x + y + xy$.
(2) $G = (\mathbf{R} \setminus \{0\}) \times \mathbf{R}$ and $(x, u) \otimes (y, v) = (xy, xv + u)$.
(3) $G = \{(x, y) \in \mathbf{R}^2,\ x^2 + y^2 = 1\}$ and $(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$.

A.2. Let G be a finite group and let $a, b \in G$ be such that $(ab)^m = e$ for some $m \geq 1$. Prove that $(ba)^m = e$.

A.3. Let G be a group such that for every $a \in G$, we have $a^2 = e$. Prove that G is abelian.

A.4.

(1) Find all groups G which have no non-trivial automorphisms.
(2) Find all groups G which have no non-trivial subgroups.

A.5. Find all continuous homomorphisms $f : (\mathbf{R}, +) \to (\mathbf{R}, +)$.

B. Problems graded by Inna

B.1. Prove that if G is an infinite group, then it has an infinite number of subgroups.

B.2. Prove that if H is a subgroup of index 2 of a group G, then H is a normal subgroup of G.

B.3. The center $Z = Z(G)$ of a group G is the set of elements $z \in G$ such that $zg = gz$ for every $g \in G$.

(1) Prove that Z is a subgroup of G.
(2) Prove that if G/Z is cyclic, then G is abelian (and so $G = Z$).
(3) Prove that if G is not abelian, then $[G : Z] \geq 4$.
(4) Define $C(G) = \{(x, y) \in G^2,\ xy = yx\}$. Prove that if G is finite and not abelian, then
$$\#C(G) \leq \frac{5}{8}(\#G)^2.$$

B.4. Let $S_4 = \mathrm{Bij}(\{a, b, c, d\})$. Prove that S_4 has a unique subgroup of order 12. Call it A_4. Prove that A_4 has no subgroup of order 6.

B.5. Prove that if H is a subgroup of a finite group G (and $H \neq G$), then $G \neq \cup_{g \in G}(gHg^{-1})$.