HOMEWORK 7 — DUE NOV 7TH

MATH 25

There are two sections: A and B. Please return each part to the proper CA.

A. Problems graded by Benjamin

A.1. Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0 \end{cases}$$

is differentiable everywhere on $\mathbf{R}$ but that f is not C^1.

A.2. Suppose that $\{f_n\}_{n \geq 1}$ is a sequence of C^1 functions $f_n : [0,1] \to \mathbf{R}$, and that $f, g : [0,1] \to \mathbf{R}$ are two functions such that: for all $x \in [0,1]$, $f_n(x) \to f(x)$ (i.e. $f_n \to f$ pointwise), and $f_n' \to g$ uniformly on $[0,1]$.

Prove that f is C^1, that $f' = g$ and that $f_n \to f$ uniformly.

B. Problems graded by Inna

B.1. Let $f : [0,1] \to \mathbf{R}$ be a continuous function. Find the limit as $n \to \infty$ of

$$I_n = n \int_0^1 x^n f(x) dx$$

and then find the next term in the asymptotic development (in order to do this, you can assume that f is as regular as you want, for example that f is C^∞).

B.2. For $n \geq 0$, let $J_n : \mathbf{R} \to \mathbf{R}$ be the n-th Bessel function, defined by

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z \sin\theta) d\theta.$$

The following questions are largely independant:

(1) Explain why $J_n(z)$ is a C^∞ function of z.
(2) Prove that $J_1(z) = -J_0'(z)$ and that $J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z)$.
(3) (extra credit) Prove that

$$J_n''(z) + \frac{1}{z} J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0.$$

(4) (extra credit) Prove that $J_n(x) \to 0$ as $x \to +\infty$.