HOMEWORK 6 — DUE OCT 31ST

MATH 25

There are two sections: A and B. Please return each part to the proper CA. Remember that you are *not* allowed to use a calculator nor a computer (in problem B.4 for example!)

A. Problems graded by Benjamin

A.1. **Filling holes.** Let f, g be two continuous functions on the interval $[a,b]$ and fix $c \in (a,b)$. Suppose that f is C^1 on $[a,b] \setminus \{c\}$ and that $f' = g$ on $[a,b] \setminus \{c\}$.

(1) Prove that f is differentiable at c and that $f' = g$ on $[a,b]$. Hint: consider the function $x \mapsto f(x) - \int_c^x g(t)dt$.
(2) Prove that the function f defined by $f(0) = 1$ and $f(x) = x/\sin(x)$ is C^1 on $[-\pi/2, \pi/2]$.

A.2. **Some Riemann sums I.** Let $I = [0,1]$ and let $f : I \to \mathbf{R}$ be a continuous function. For $n \geq 1$, define a step function f_n by:

$$f_n(x) = \begin{cases} f(1/n) & \text{if } 0 \leq x \leq 1/n \\ f(k/n) & \text{if } 1 \leq k \leq n-1 \text{ and } k/n < x \leq (k+1)/n. \end{cases}$$

(1) Prove that $|f_n - f|_I \to 0$ as $n \to +\infty$;
(2) use this to show that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f$$

(3) prove that the above formula still holds if f is a ruled function on I.

A.3. **Some Riemann sums II.** Using the results of the previous exercise, compute the following limits:

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{k+n}, \quad \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}, \quad \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{k^3}{n^4} \cos\left(\frac{4k}{n}\right).$$

A.4. **Some Riemann sums III.** Let n be an integer ≥ 2. If $1 \leq i \leq n$, we write $n = q_{i,n}i + r_{i,n}$ with $0 \leq r_{i,n} \leq i-1$. Let a_n be the number of i's with $1 \leq i \leq n$ such that $r_{i,n} \geq i/2$. Let $\phi : I \to \mathbf{R}$ be defined by $\phi(0) = 0$ and $\phi(x) = \lfloor 2/x \rfloor - 2\lfloor 1/x \rfloor$, where $\lfloor y \rfloor$ denotes the integer part of y.

(1) prove that $q_{i,n}$ and $r_{i,n}$ above are uniquely determined;
(2) prove that ϕ is a ruled function;
(3) compute $\int_{1/k}^{1} \phi$ if $k \geq 2$ and use this to compute $\int_0^1 \phi$;
(4) prove that $a_n = \sum_{k=1}^{n} \phi(k/n)$;
(5) prove that $\lim_{n \to +\infty} a_n/n$ exists and compute it.

B. Problems graded by Inna

B.1. **Euler's constant.**

(1) prove that there exists a constant $M \geq 0$ such that $0 \leq x - \log(1+x) \leq Mx^2$ if $0 \leq x \leq 1$;
(2) use this to show that

$$\lim_{n \to +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \log(n)$$

exists. It is called Euler's constant.

B.2. **Cauchy-Schwarz' inequality.** Let $f, g : [a,b] \to \mathbf{R}$ be two ruled functions. Prove that

$$\left(\int_a^b fg \right)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

Hint: obviously, $\int_a^b (f - gt)^2 \geq 0$. Now judiciously choose t.

B.3. **Four inequalities.**

(1) Prove that if $f, g : [0,1] \to \mathbf{R}$ are non-decreasing ruled functions, then

$$\int_0^1 fg \geq \left(\int_0^1 f \right) \left(\int_0^1 g \right).$$

(2) prove that if $f : [0,1] \to \mathbf{R} \setminus \{0\}$ is a continuous function, then

$$\left(\int_0^1 f \right) \left(\int_0^1 1/f \right) \geq 1.$$

(3) Let $f : [0,1] \to \mathbf{R}$ be a C^1 function such that $f(0) = f(1) = 0$. Prove that

$$\left| \int_0^1 f \right| \leq \frac{1}{4} |f'|_{[0,1]}.$$

Can one replace $1/4$ by a smaller number?

(4) Let $f : [0, \pi] \to \mathbf{R}$ be a C^1 function such that $f(0) = f(\pi) = 0$. Prove that
$$\int_0^\pi f(t) f'(t) \operatorname{cotan}(t) dt = \frac{1}{2} \int_0^\pi f^2(t)(1 + \operatorname{cotan}^2(t)) dt.$$
and use this to show that
$$\int_0^\pi f^2 \leq \int_0^\pi (f')^2.$$

B.4. **Lots of derivatives.** Recall that $\sinh(x) = (e^x - e^{-x})/2$. Let $f(x) = \sin(\sinh(x)) - \sinh(\sin(x))$. Find the least integer $n \geq 0$ such that $f^{(n)}(0) \neq 0$ and compute $f^{(n)}(0)$. Hint: you can use Taylor's expansion for sin and sinh.