HOMEWORK 4 — DUE OCT 17TH

MATH 25

There are three sections: A,B and C (plus a fourth "optional" one which won't be graded). Please return the three parts *separately* to the proper CA.

A. Problems graded by Jennifer

A.1. **A regularization operator.** Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Set

$$y_n = \frac{1}{n}\left(x_1 + x_2 + \cdots + x_n\right)$$

If $y_n \to y$ for some $y \in \mathbf{R}$, we say that $\{x_n\}_{n\geq 1}$ Cesàro-converges to y.

(1) Show that if x_n converges to x, then it also Cesàro-converges to x.
(2) Find the Cesàro-limit of the sequence $\{x_n = (-1)^n\}$.

A.2. **Extending continuous functions.** Let $f : [0,1) \to \mathbf{R}$ be a continuous function. Show that one can extend f to a continuous function on $[0,1]$ if and only if f is uniformly continuous on $[0,1)$.

A.3. **Discontinuities of non-decreasing functions.** Let $f : \mathbf{R} \to \mathbf{R}$ be a non-decreasing function.

(1) Show that f is continuous at $x \in \mathbf{R}$ if and only if $\sup_{y<x} f(y) = f(x)$ and $\inf_{y>x} f(y) = f(x)$ (draw a picture!).
(2) Prove that the set of $x \in \mathbf{R}$ at which f is not continuous is countable.

B. Problems graded by Benjamin

B.1. **Writing a number in base b.** Let $b \geq 2$ be an integer, and choose $n \in \mathbf{N}$.

(1) Show that one can write $n = \sum_{k=0}^{\ell} x_k b^k$ with $x_k \in \{0, 1, \cdots, b-1\}$ and some $\ell \geq 0$ in exactly one way.
(2) Choose $x \in [0,1)$ now. Show that one can write $x = \sum_{k=1}^{+\infty} x_k b^{-k}$ with $x_k \in \{0, 1, \cdots, b-1\}$ in exactly one way, if we impose the additional condition that for every $N \geq 0$, there exists $n \geq N$ such that $x_n \neq b-1$.

B.2. **The digits of 2^n.** Let $\log_2$ stand for the logarithm in base 2: if $x > 0$, $\log_2(x)$ is the unique real number such that $2^{\log_2(x)} = x$.

(1) Prove that $\log_2(10)$ is irrational (hint: what can you say if $2^{p/q} = 10$?).
(2) Show that there is a number $n \geq 1$ such that when one writes 2^n in base 10, the first few digits are 121 219 76....

B.3. **Lévy's chords lemma.** Let $f : [0, 1] \to \mathbf{R}$ be a continuous function such that $f(0) = f(1)$ and choose $q \in \mathbf{Z}_{\geq 1}$.

Prove that there exists $x \in [0, 1 - 1/q]$ such that $f(x + 1/q) = f(x)$.

C. Problems graded by Inna

C.1. **Lim sup and lim inf.** Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers, and set $y_m = \sup_{n\geq m} x_n$.

(1) Prove that $\{y_m\}_{m\geq 1}$ is a non-increasing sequence bounded from below, so that it converges. Define $\limsup x_n$ to be the limit of y_m;
(2) Give a similar definition of $\liminf x_n$;
(3) Prove that $\limsup x_n \geq \liminf x_n$ and that $\{x_n\}_{n\geq 1}$ has a limit if and only if $\limsup x_n = \liminf x_n$;
(4) Recall that if $A(x)$ is the set of accumulation values of $\{x_n\}_{n\geq 1}$, then $A(x)$ is closed. Why does it have a largest element? Prove that $\ell = \limsup x_n$ is the largest element of $A(x)$ and that for any $\epsilon > 0$ the set $\{n \in \mathbf{Z}_{\geq 1} \quad x_n > \ell + \epsilon\}$ is finite;
(5) Give the corresponding statements for $\liminf$.

C.2. **A special class of sequences.** Let $\{\alpha_n\}_{n\geq 1}$ be a sequence of real numbers bounded from below and such that $(n + m)\alpha_{n+m} \leq n\alpha_n + m\alpha_m$.

Show that there exists $\alpha \in \mathbf{R}$ such that $\alpha_n \to \alpha$. Hint: use the preceding exercise.

C.3. **Rearranging a series.** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, which converges to $\ell \in \mathbf{R}$. If $\phi : \mathbf{Z}_{\geq 1} \to \mathbf{Z}_{\geq 1}$ is a bijection, we consider the rearranged series $\sum_{n=1}^{\infty} a_{\phi(n)}$.

(1) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any rearrangement $\sum_{n=1}^{\infty} a_{\phi(n)}$ also converges to ℓ.
(2) Assume now that $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent Let P and N be the subsets of $\mathbf{Z}_{\geq 1}$ such that $m \in P$ if $a_m \geq 0$ and $m \in N$ if $a_m < 0$. Prove that P and N both contain infinitely many elements $\neq 0$, and that furthermore $\sum_{k\in P} a_k \to +\infty$ and $\sum_{m\in N} a_m \to -\infty$.

(3) Let z be any real number. Set $\phi(1) = 1$ and define $\phi(n)$ by recurrence in the following way: if $\sum_{k=1}^{n-1} a_{\phi(k)} > z$ then set $\phi(n)$ to be any element of N which has not already been chosen. Otherwise, set $\phi(n)$ to be any element of P which has not already been chosen.

(4) Prove that the rearranged series $\sum_{n=1}^{\infty} a_{\phi(n)}$ converges to z.

D. Supplementary problems

D.1. **Dilating maps.** Let (E, d) be a compact metric space and let $f : E \to E$ be a continuous map such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in E$. Show that f is a bijection and that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in E$.

D.2. **Steinitz' theorem.** Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers such that $\sum_{n\geq 1}(a_n, b_n) = (0, 0)$. Let S be the set of all possible limits in $\mathbf{R}^2$ of $\sum_{n\geq 1}(a_{\phi(n)}, b_{\phi(n)})$ when ϕ runs among all bijections $\phi : \mathbf{Z}_{\geq 1} \to \mathbf{Z}_{\geq 1}$. Show that S is either $\{(0, 0)\}$ or a line or $\mathbf{R}^2$ itself. Generalize to higher dimensions.