HOMEWORK 3 — DUE OCT 10TH

MATH 25

There are three sections: A,B and C (plus a fourth "optional" one which won't be graded). Please return the three parts *separately* to the proper CA.

A. Problems graded by Jennifer

A.1. **Distance to a subset.** Let (E, d) be a metric space, and let A be a subset of E. For $x \in E$, let $d(x, A) = \inf_{a \in A} d(x, a)$.

(1) Show that $x \mapsto d(x, A)$ is continuous;
(2) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

A.2. **Extrema on compact spaces.** Show that if E is a compact metric space and $f : E \to \mathbf{R}$ is a continuous map, then f admits a maximum on E. That is, there exists $x \in E$ such that $f(x) \geq f(y)$ for any $y \in E$.

Use this to show that if F_1, F_2 are two disjoint closed subsets of E then there exists $\mu > 0$ such that $d(a_1, a_2) \geq \mu$ whenever $a_1 \in F_1$ and $a_2 \in F_2$.

A.3. **Contracting maps.** Let (E, d) be a compact metric space and let $f : E \to E$ be a continuous map such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in E$ with $x \neq y$.

Define the diameter of a closed subset F of E to be $\operatorname{diam}(F) = \sup_{x,y \in F} d(x, y)$. What are the closed subsets of E whose diameter is 0?

(1) Show that there exist $x, y \in F$ such that $\operatorname{diam}(F) = d(x, y)$;
(2) Show that $\operatorname{diam}(f(F)) < \operatorname{diam}(F)$ if $\operatorname{diam}(F) \neq 0$;
(3) Use this to show that f has a unique fixed point.

A.4. **Accumulation points II.** Let E be a compact metric space, let $u = \{u_n\}_n$ be a sequence of elements of E, and let $A(u)$ be the set of its accumulation points. Show that if $d(u_n, u_{n+1}) \to 0$ as $n \to +\infty$ then $A(u)$ is connected.

Hint: if you can write a space X as the disjoint union of two open sets, then those open sets are also closed.

B. Problems graded by Benjamin

B.1. **Uniform continuity.**

(1) show that if $f : \mathbf{R}_{\geq 0} \to \mathbf{R}$ is continuous and $f(x) \to \ell$ when $x \to +\infty$, then f is uniformly continuous;
(2) show that if $f : \mathbf{R}_{\geq 0} \to \mathbf{R}$ is uniformly continuous then there exist $A, B \in \mathbf{R}$ such that $|f(x)| \leq Ax + B$;
(3) Which of the following functions from $\mathbf{R}_{\geq 0}$ to $\mathbf{R}$ are uniformly continuous?

$$x \mapsto e^x \qquad x \mapsto \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad x \mapsto \sqrt{x}.$$

B.2. Show that if F is a closed subset of $[0,1]$ then there exists a continuous function $f : [0,1] \to \mathbf{R}$ such that $f^{-1}(0) = F$.

B.3. Show that if $f : \mathbf{R} \to \mathbf{R}$ is bijective and continuous, then f is a homeomorphism (this is obvious if you draw a picture, but you need to give a rigorous proof).

C. Problems graded by Inna

C.1. Let E be a compact metric space, and for $n \geq 1$ let F_n be a closed subset of E, such that F_n is non-empty and $F_{n+1} \subset F_n$. Show that $\cap_{n=1}^{+\infty} F_n$ is non-empty.

Give an example to show that this can fail if E is not compact.

C.2. Show that if E is compact, then it contains a dense denumerable subset (for example, $[0,1]$ contains $[0,1] \cap \mathbf{Q}$).

C.3. **The Cantor Set.** If $I \subset [0,1]$ is a finite union of disjoint closed intervals $[a_i, b_i]$, let $T(I)$ be the union of the intervals $[a_i, a_i + (b_i - a_i)/3]$ and $[a_i + 2(b_i - a_i)/3, b_i]$ (one "removes the middle third" from each interval in I to get $T(I)$).

Let $K_0 = [0,1]$ and $K_n = T(K_{n-1})$ for $n \geq 1$. Define $K = \cap_{n=0}^{+\infty} K_n$.

(1) Show that K is compact;
(2) Show that $[0,1] \setminus K$ is a coutable union of intervals the sum of whose lengths is 1;
(3) Show that the connected component of each point x of K is the point x itself;
(4) Show that K is also the set of $\alpha \in [0,1]$ such that one can write $\alpha = \sum_{i=0}^{+\infty} \alpha_i 3^{-i}$ with $\alpha_i \in \{0, 2\}$ for every i;
(5) Show that K is not countable.

One can also show that if X is *any* compact metric space, then there is a surjective map $f : K \to X$ so that K is some "universal" compact space.

D. Optional problems

We will most likely prove the following theorems in class later in the year, but you can try your hand at them now. You can also try proving the assertion above, that if X is any compact metric space, then there is a surjective map $f : K \to X$.

D.1. **Borel-Lebesgue's theorem.** Let E be a metric space. Show that the following three properties are equivalent:

(1) E is compact;
(2) if $\{U_i\}_{i\in I}$ is any family of open subsets of E such that $E = \cup_{i\in I} U_i$ then there exists a finite subset $J \subset I$ such that $E = \cup_{j\in J} U_j$;
(3) if $\{F_i\}_{i\in I}$ is any family of closed subsets of E such that $\cap_{i\in I} F_i = \varnothing$ then there exists a finite subset $J \subset I$ such that $\cap_{j\in J} F_j = \varnothing$.

D.2. **Baire's theorem.** Let E be a complete metric space and let $\{U_i\}_{i\in I}$ be a countable family of dense open subsets of E. Show that $\cap_{i\in I} U_i$ is still dense in E (in general it won't be open!).

Show that this can fail if I is not countable.