HOMEWORK 2 — DUE OCT 3RD

MATH 25

There are three sections: A,B and C (plus a fourth "optional" one which won't be graded). Please return the three parts *separately* to the proper CA.

A. Problems graded by Jennifer

A.1. Give an example of a metric space (E, d) and infinitely many open sets $\{U_i \subset E\}_{i\in I}$ such that $\cap_{i\in I} U_i$ is not open. You could consider suitable intervals in $\mathbf{R}$ for example.

A.2. Let X be a subset of a metric space (E, d). Show that:

$$\mathring{\overline{X}} = \mathring{\overline{\mathring{\overline{X}}}} \quad \text{and that} \quad \overline{\mathring{X}} = \overline{\mathring{\overline{\mathring{X}}}}.$$

Give an example of a subset X of $\mathbf{R}$ such that the following sets:

$$X,\ \overline{X},\ \mathring{X},\ \mathring{\overline{X}},\ \overline{\mathring{X}},\ \overline{\mathring{\overline{X}}},\ \mathring{\overline{\mathring{X}}},$$

are pairwise distinct.

A.3. **Induced topology.** Let (E, d) be a metric space, and let X be a subset of E so that (X, d) is itself a topological space with the induced distance. Show that $U \subset X$ is open in X if and only if there exists an open set V of E such that $U = X \cap V$.

Be careful: if $U \subset X$ is open in X, it is not necessarily open in E.

A.4. **Product topology.** Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Define a distance d on $E_1 \times E_2$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

(1) Show that $U \subset E_1 \times E_2$ is open if and only if for each point $(u_1, u_2) \in U$, there exist U_1 and U_2 open in E_1 and E_2 with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$.
(2) If X is another metric space, show that $(f_1, f_2) : X \to E_1 \times E_2$ is continuous if and only if f_1 and f_2 both are.

B. Problems graded by Benjamin

B.1. **The p-adic topology.** Given $a \in \mathbf{Q}^*$ there is a unique $v = v(a) \in \mathbf{Z}$ such that one can write $a = p^v n/d$ with $n, d \in \mathbf{Z}$ relatively prime to p. For $x, y \in \mathbf{Q}$ define $d_p(x, y) = p^{-v(x-y)}$ if $x \neq y$ and $d(x, x) = 0$.

(1) Show that $(\mathbf{Q}, d_p)$ is a metric space and that actually $d(x, z) \leq \max(d(x, y), d(y, z))$;
(2) show that if $x \in B(a, r)$, then $B(x, r) = B(a, r)$ so that any point of the ball $B(a, r)$ is a "center" of that ball;
(3) show that given two balls $B(a_1, r_1)$ and $B(a_2, r_2)$, then either they're disjoint or one is contained in the other;

This goes to show that metric spaces can be quite strange.

B.2. **Accumulation points.** If $u = \{u_n\}_{n\geq 1}$ is a sequence of elements of a metric space (E, d), let $A(u)$ be the set of its accumulation points. Show that $A(u)$ is closed.

B.3. Let $f : [0, 1] \to \mathbf{R}$ be defined by $f(x) = \sqrt{x}$. Show that f is uniformly continuous.

B.4. We say that a function $f : X \to Y$ is K-Lipshitz if $d_Y(f(x), f(x')) \leq K \cdot d_X(x, x')$ for all $x, x' \in X$. Check that such a function is uniformly continuous.

Show that for every $0 < a < 1$, there exists K_a such that $f(x) = \sqrt{x}$ is K_a-Lipshitz on $[a, 1]$. What happens if $a = 0$?

C. Problems graded by Inna

C.1. We say that a function $f : X \to Y$ is continuous at a point x if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, x') < \delta$, then $d(f(x), f(x')) < \epsilon$. This way, $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.

(1) Show that f is continuous at x if and only if for every sequence $\{x_n\}_n$ which converges to x, we have $f(x_n) \to f(x)$;
(2) If $x \in \mathbf{Q}$, one can write $x = p/q$ in a unique way with $\gcd(p, q) = 1$ and $q \geq 1$ (if $x = 0$, set $p = 0$ and $q = 1$). Let $f : [0, 1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbf{R} \setminus \mathbf{Q}, \\ 1/q & \text{if } x = p/q \in \mathbf{Q}. \end{cases}$$

At what points x of $[0, 1]$ is f continuous?

C.2. Let $\mathbf{R}^2$ be the plane with the usual distance and let $x_1, \cdots, x_n$ be finitely many points. Show that $\mathbf{R}^2 \setminus \{x_1, \cdots, x_n\}$ is connected.

C.3. This problem has been removed and will be in the third set.

C.4. Show that a circle is not homeomorphic to an interval of $\mathbf{R}$. Show also that a "figure eight" is not homeomorphic to a circle.

Hint: think about connectedness.

D. Supplementary problems

If you get bored, do the following.

D.1. In exercise B.1, show that $(\mathbf{Q}, d_p)$ is not complete.

D.2. Show that if F is a closed subset of a metric space E and if $f : F \to [0,1]$ is a continuous function, then there exists a continuous function $\tilde{f} : E \to [0,1]$ which coincides with f on F.