FINAL EXAM

MATH 25

INSTRUCTIONS: READ CAREFULLY

This exam starts at 9:15 and finishes at 12:15.

- You **are** allowed to use: your notes from class, anything that's on the course's website or on the CA's website. It is your responsibility to check that what you're using is indeed mathematically correct.
- You **are not** allowed to use: (text)books, photocopies, a calculator or a computer.

You can use scrap paper but I will only grade what's in the "blue books".

Name:____

Return this sheet with your solutions.

§	1.1	1.2	1.3	2.1	2.2	3.1	3.2	3.3	3.4	3.5	Total
Points											
Out of	10	10	10	10	10	10	10	10	10	10	100

1. SIMULTANEOUS DIAGONALISATION

Let f and g be two endomorphisms of a d-dimensional vector space V over a field K. We say that f and g are *simultaneously diagonalisable* if there exists a basis $e_1, \cdots, e_d$ of V and elements $\{\lambda_i\}_{i=1\cdots d}$ and $\{\mu_i\}_{i=1\cdots d}$ of K such that $f(e_i) = \lambda_i e_i$ and $g(e_i) = \mu_i e_i$ for $i = 1 \cdots d$.

Observe that if f and g are simultaneously diagonalisable, then $fg = gf$. In the rest of this problem, we will assume that f and g are diagonalisable and that $fg = gf$. The goal of the problem is to prove that this implies that f and g are simultaneously diagonalisable.

1.1. **10 pts.** Prove that if W is a sub-vector space of V, stable under f, then the restriction of f to W is diagonalisable.

1.2. **10 pts.** Prove that if $\mu \in K$, then the vector space $\ker(g - \mu\,\mathrm{Id})$ is stable under f.

1.3. **10 pts.** Prove that f and g are simultaneously diagonalisable.

2. Topology of some spaces of matrices

In the remainder of this exam we will study the topology of certain spaces of matrices with complex coefficients. The absolute value $|\cdot|$ is the usual one on $\mathbf{C}$. We will need to use three topological spaces:

- The space $\mathbf{C}^n$ is the space of n-uples $(x_1, \cdots, x_n)$ with the distance defined by $d(x,y) = \sup_{1\leq i\leq n} |x_i - y_i|$ if $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.
- The space $\mathbf{C}[X]_n$ is the set of polynomials $P(X) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$ with the distance defined by $d(P,Q) = \sup_{0\leq i\leq n-1} |p_i - q_i|$ if $P(X) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$ and $Q(X) = X^n + q_{n-1}X^{n-1} + \cdots + q_0$.
- The space $\mathrm{M}(n, \mathbf{C})$ is the space of all $n \times n$ matrices with complex coefficients. If $A = (a_{ij})_{1\leq i,j\leq n}$ and $B = (b_{ij})_{1\leq i,j\leq n}$, we define $d(A,B) = \sup_{1\leq i,j\leq n} |a_{ij} - b_{ij}|$.

2.1. **10 pts.** Explain *briefly* why the following maps are continuous. You should not use ϵ/δ arguments; no more than a few lines of justification are necessary:

(1) $\det : \mathrm{M}(n, \mathbf{C}) \to \mathbf{C}$.
(2) the "characteristic polynomial" map $\chi : \mathrm{M}(n, \mathbf{C}) \to \mathbf{C}[X]_n$ which maps M to $\chi_M = \det(X \operatorname{Id} - M)$.
(3) the map $f : \mathbf{C}^n \to \mathbf{C}[X]_n$ which maps $(z_1, \cdots, z_n)$ to $\prod_{i=1}^n (X - z_i)$.
(4) the "conjugation by P" map $M \mapsto PMP^{-1}$ from $\mathrm{M}(n, \mathbf{C})$ to itself, where P is an invertible matrix.

2.2. **10 pts.** Prove that the set of invertible matrices $M \in \mathrm{M}(n, \mathbf{C})$ is open in $\mathrm{M}(n, \mathbf{C})$.

3. Diagonalisable matrices

Let $\mathrm{D}(n, \mathbf{C})$ be the subset of $\mathrm{M}(n, \mathbf{C})$ consisting of diagonalisable matrices. We will show that $\mathrm{D}(n, \mathbf{C})$ is dense in $\mathrm{M}(n, \mathbf{C})$ and we will compute its interior.

3.1. **10 pts.** Let M be a matrix which is upper triangular. Prove that each diagonal entry is an eigenvalue of M.

3.2. **10 pts.** Prove that $\mathrm{D}(n, \mathbf{C})$ is dense in $\mathrm{M}(n, \mathbf{C})$.

3.3. **10 pts.** Prove that if E is a non-empty subset of $\mathbf{C}^n$ then E is bounded if and only if $f(E) \subset \mathbf{C}[X]_n$ is bounded (here f is the map from (3) of exercise 2.1).

3.4. **10 pts.** Use this to prove that the set of polynomials having only simple roots is open in $\mathbf{C}[X]_n$.

3.5. **10 pts.** Prove that the interior of $\mathrm{D}(n, \mathbf{C})$ in $\mathrm{M}(n, \mathbf{C})$ is the set of matrices having n distinct eigenvalues.