MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 30, supplement

Alternating and Skew-Symmetric Forms

**Definition:** A multilinear form $f : V^k \longrightarrow F$ is said to be *alternating* if and only if:

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k) = 0$$

whenever $\mathbf{v}_i = \mathbf{v}_j$ with $i \neq j$.

**Definition:** A multilinear form $f : V^k \longrightarrow F$ is said to be *skew-symmetric* if and only if:

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k) = -f(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k)$$

for every pair $i \neq j$.

**Theorem:** If the multilinear form $f : V^k \longrightarrow F$ is alternating, then it is skew-symmetric.

> **Proof:** For simplicity of notation but without any loss of generality, we give the proof for $k = 2$. Note that since $f$ is alternating, we use the fact that $f(\mathbf{v}, \mathbf{v}) = 0$ for any $\mathbf{v} \in V$ in several of the steps below:
>
> $$\begin{aligned} f(\mathbf{v}_1, \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1) &= [f(\mathbf{v}_1, \mathbf{v}_2) + f(\mathbf{v}_1, \mathbf{v}_1)] + [f(\mathbf{v}_2, \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1)] \\ &= f(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\ &= f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\ &= 0 \end{aligned}$$
>
> This implies that $f(\mathbf{v}_1, \mathbf{v}_2) = -f(\mathbf{v}_2, \mathbf{v}_1)$.