If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

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We associate $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

1. Let $A = (a_{ij})$. We are interested in $\nabla(\det(A))$. We can pick a row and expand by minors:

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}),$$

where A_{ij} is the minor, taken by crossing out the ith row and jth column of the matrix.

Using this expression for the determinant, we can take the partial with respect to a_{ij}. None of the minors contain a_{ij}, so the determinant of the minor is a constant. Hence,

$$\partial \det(A)/\partial a_{ij} = (-1)^{i+j} \det(A_{ij}).$$

So,

$$\nabla(\det(A)) = (\ldots, (-1)^{i+j} \det(A_{ij}), \ldots).$$

2. Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $f(A) = A^2$. First, note that $f(A)$ is a second-degree polynomial in n^2 variables, the coefficients a_{ij} of A. Since polynomials are differentiable, f is differentiable.

This implies there exists a unique linear map L such that

$$\lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - L(H)}{\|H\|} = 0.$$

Let us verify that $L = AH + HA$ works. We have,

$$\begin{aligned} \lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - L(H)}{\|H\|} &= \lim_{\|H\| \to 0} \frac{A^2 + AH + HA + H^2 - A^2 - AH - HA}{\|H\|} \\ &= \lim_{\|H\| \to 0} \frac{H^2}{\|H\|}. \end{aligned}$$

Now, consider

$$\begin{aligned} \left\| \lim_{\|H\| \to 0} \frac{H^2}{\|H\|} \right\| &= \lim_{\|H\| \to 0} \frac{\|H\| \, \|H\|}{\|H\|} \\ &= \lim_{\|H\| \to 0} \|H\| = 0. \end{aligned}$$

Since 0 is the only number whose norm is 0, this proves the result.

Philosophically, this is just saying that the derivative of x^2 is $x + x = 2x$, but in a non-commutative setting (matrix multiplication).