MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 3
Due: February 21, 2003

Homework Assignment #3 (Final Version)

1. Read Fitzpatrick, Sections 13.3, 15.2, and 15.3.

2. (A) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Show that:

   (a) $\nabla(f+g) = \nabla f + \nabla g$.

   (b) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.

   (c) $\nabla(f^m) = mf^{m-1}\nabla f$, for any positive integer m.

   (d) Determine a formula for $\nabla(\frac{f}{g})$ when $g(\mathbf{x}) \neq 0$.

3. (deferred) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, Theorem 13.3 applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

   With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ the usual gradient of f, we make the following definitions:

   - $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f.
   - $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the Laplacian of f.

   Finally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $g(r,\theta) = f(r\cos\theta, r\sin\theta)$.

   (a) Show that $||\nabla f||^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.

   (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$.

4. (B) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

   (a) (Not required) Graph $z = f(x,y)$ using Mathematica.

   (b) Show that $D_1 f(0,y) = -y$ and $D_2 f(x,0) = x$ for all $x, y \in \mathbb{R}$.

   (c) Show that $D_2 D_1 f(0,0)$ and $D_1 D_2 f(0,0)$ exist but are not equal.

5. (C) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable and has an inverse function $f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is also differentiable. Show that:

$$[J(f^{-1})](\mathbf{a}) = [(Jf)(f^{-1}(\mathbf{a}))]^{-1}.$$

Sorry for all the parentheses, but I am trying to make this clear. On the left-hand side, we are taking the Jacobian of f^{-1} *and evaluating at* $\mathbf{a}$*. On the right-hand side, we are taking the Jacobian of* f *and evaluating at* $f^{-1}(\mathbf{a})$*, and then taking the inverse (as a matrix) of that.*

6. (D) Recall that we have an isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

   (a) Consider the determinant map $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, and find $\nabla(\det)(A)$, expressed in terms of $A = [a_{ij}]$.

   (b) Consider the function $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.

7. (E) Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $f(\mathbf{x}) = ||\mathbf{x}||\mathbf{x}$. Determine whether or not f is differentiable at $\mathbf{0}$. If not, why not? If so, find the first-order partial derivatives of f at $\mathbf{0}$. (Bonus: Do the second-order partial derivatives of f exist at $\mathbf{0}$.?)