Math 23a Solution: Problem A

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We are looking for a linear bijection between the vector spaces $C[0,1]$ and $C[a,b]$ for any $a < b$. To start, let F be a continuous bijection from $[a,b]$ to $[0,1]$ whose inverse is also continuous (the easiest example–and the one everyone used–is $F(x) = (x-a)/(b-a)$, which has inverse $G(y) = (b-a)y + a$, both of which are clearly continuous). Now define a map $F^* : C[0,1] \to C[a,b]$ by $F^*(f) = f \circ F$ for $f \in C[0,1]$. This is well defined, since the range of F is equal to the domain of f, and in fact is continuous, with $[a,b]$ as its domain. Thus $f \circ F \in C[a,b]$, as desired. Note that even though F maps $[a,b]$ to $[0,1]$, F^* maps $C[0,1]$ to $C[a,b]$; F^* goes in the opposite direction from F. This is a subtle point, and one responsible for most of the points lost on this problem.

Linearity is a consequence of simple properties of function composition:

$$F^*(af+bg) = (af+bg)\circ F = (af)\circ F+(bg)\circ F = a(f\circ F)+b(g\circ G) = aF^*(f)+bF^*(g)$$

We will find an inverse for F^*, thus showing that it is invertible. We claim that $G^* : C[a,b] \to C[0,1]$ defined by $G^*(g) = g \circ G$ is the inverse of F^*. G^* is well defined and linear by the same arguments for F^*. If $f \in C[0,1]$, we have $G^*(F^*(f)) = G^*(f \circ F) = f \circ F \circ G = f$ since $F \circ G$ is the identity function on $[0,1]$. Similarly, $F^*(G^*(g)) = g$ for $g \in C[a,b]$, and thus G^* is the inverse of F^*. This proves that F^* is an isomorphism.