If V is a vector space $B = \{e_1, ..., e_d\}$ be a basis of V and $T : V \to V$ is a linear transformation we denote by $A_B(T)$ the matrix of T in the basis B. That is $A_B(T) = (a^i_j), 1 \leq i, j \leq d$ is a matrix such that $T(e_j) = \sum_{i=1}^d a^i_j e_i$.

1. a)Let $B_0 =: \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2, T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$A_{B_0}(T) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

[that is

$$T(e_1) = 2e_1 + e_2, T(e_2) = e_1 + e_2]$$

Find a basis $B =: \{f_1, f_2\}$ of $\mathbb{R}^2$ such that the matrix $A_B(T)$ is diagonal. That is $A_B(T) = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

b)find the numbers λ_1 and λ_2.

c)Let $f_1 = c^1_1 e_1 + c^2_1 e_2, f_2 = c^1_2 e_1 + c^2_2 e_2, c^i_j \in \mathbb{R}$.

Show that $A = C\Lambda C^{-1}$ where $A := A_{B_0}(T)$.

$C = \begin{pmatrix} c^1_1 & c^2_1 \\ c^1_2 & c^2_2 \end{pmatrix}$

d)Let $a = (3 - \sqrt{5})/2$. Find the elements of the matrix $a^{1000} A^{1000}$ with the precision of 10 percent. "With the precision of 10 percent" means that if e is the exact (unknown) answer and a is the approximate answer which you give, you must have $|a - e|/a < 0.1$.

2.a) Let f be a differentiable function on $\mathbb{R}$ such that $f'(x) \equiv 1$.

Prove that there exists $c \in \mathbb{R}$ such that $f(x) = x + c$ for all $x \in \mathbb{R}$.

b)Find all twice differetiable functions F on $\mathbb{R}$ such that $F''(x) \equiv 1$.

Let $f : Mat(n, n) \to Mat(n, n)$ be a function such that $f(A) = A^2$.
3. a)Show that the function f is differentiable and for any $n \times n$ matrix H find the differential $df(H) : Mat(n, n) \to Mat(n, n)$.

We can identify the vector space $Mat(n, n)$ with $\mathbb{R}^{n^2}, A \to \{a_{i,j}\}$. Therefore we can define the partial derivatives $\partial f / \partial a_{i,j}$.

b) Find the partial derivatives $\partial f / \partial a_{i,j}$.

c) Find the differential $dF(H) : Mat(n, n) \to Mat(n, n)$ where $F(A) := A^3$.

4. a) Let f be a differentiable function on $\mathbb{R}$ and F be and function on $\mathbb{R}^2$ defined by $F(x, y) := x^7 f(y/x^2)$.

Show that $x\partial F/\partial x + 2y\partial F/\partial y = 7F$

b) Find a generalization of this result.

If f is an infinitely differetiable function on $\mathbb{R}$ at $a \in \mathbb{R}$ we can define the Taylor series

$$P_f(a,x) := \sum_{n \geq 0} c_n (x-a)^n, c_n := f^{(n)}(a)/n!$$

. We define by $I_a \subset [0, \infty)$ as the set of numbers $r \geq 0$ such that $c_n r^n \to 0$ for $n \to \infty$ and write $r_f(a) := lub(I_a)$. [It could be that $r_f(a) = \infty$].

Let f be a function on $\mathbb{R} - \{0,1,2\}$ given by $f(x) := \frac{1}{x(x-1)(x-2)}$.

5. a)Show that f is an infinitely differetiable function on $\mathbb{R} - \{0,1,2\}$

b)Find $r_f(a)$ for any $a \in \mathbb{R}, a \neq 0,1,2$

Let V be a vector space, $\mathcal{Q}(V)$ the space of quadratic functions on V and $\mathcal{B}^2_s(V)$ be the space of symmetric biliner forms on V. To any symmetric biliner form f on V we associate a function $q(f)$ on V where $q(f)(v) := f(v,v)$.

6. a) Show that for any $f \in \mathcal{B}^2_s(V)$ the function $q(f)$ on V is quadratic.

Part a) shows that we have defined a map $\alpha : \mathcal{B}^2_s(V) \to \mathcal{Q}(V), \alpha(f) := q(f)$.

b)Construct explicitely a linear map $\beta : \mathcal{Q}(V) \to \mathcal{B}^2_s(V)$ which is the inverse of α. (Note that it follows from this that α gives an isomorphism between the vector spaces $\mathcal{B}^2_s(V)$ and $\mathcal{Q}(V)$.)

c)Suggest a generalization of of the construction $f \to q(f)$ and try to generalize the problems a)-b).

7. Let f be a twice differentiable function on $\mathbb{R}^2$ and $g : \mathbb{R}^2 - \{x,0\} \to \mathbb{R}^2$ be given by $g(x,y) := (xy, x/y)$.

Find first and second partial derivatives of the function $F := f \circ g$ in terms of the partial derivatives of the function f