MATH 23A 1ST MIDTERM SOLUTIONS

1. (George) There are slicker ways of approaching the problem, but just for your amusement, I will multiply matrices.

$$AB = BA$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$
$$\begin{bmatrix} a+d & b+e & c+f \\ d+g & e+h & f+i \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & a+b & b+c \\ d & d+e & e+f \\ g & g+h & h+i \end{bmatrix}$$

This clearly implies that $d = g = h = 0$, $a = e = i$, and $b = f$. Furthermore, it is clear that when these conditions are satisfied $AB = BA$. Therefore, the set of such B is the set of matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$.

2. (David) Let $V = \mathbb{R}^3$, $\{e_1, e_2, e_3\}$ be the standard basis of V, $v_1 = e_1 + e_2, v_2 = e_2 + e_3, v_3 = e_3$. Let $\{e^1, e^2, e^3\}$, $\{v^1, v^2, v^3\} \in V^\vee$ be the dual bases. Write v^2 as a linear combination of e^1, e^2, e^3.

   We know that $\{e^1, e^2, e^3\}$ is a basis of $V^\vee$. Therefore we can find real numbers a, b, c such that $v^2 = ae^1 + be^2 + ce^3$. By the definition the vector v^2 is the unique linear functional on V such that $v^2(v_1) = v^2(v_3) = 0$ and $v^2(v_2) = 1$. By the definiton we have $v^2(v_1) = (ae^1 + be^2 + ce^3)(e_1 + e_2) = a + b$, $v^2(v_2) = (ae^1 + be^2 + ce^3)(e_2 + e_3) = b + c$ and $v^2(v_3) = (ae^1 + be^2 + ce^3)(e_3) = c$. So we have to solve the following system of equations

$$a + b = 0$$

,

$$b + c = 1$$

and

$$c = 0$$

. This gives $a = -1, b = 1, c = 0$

   So $v^2 = -e^1 + e^2$.

3. (Li-Chung)
   (a) Define addition as $(a + bi) + (c + di) = (a + c) + (b + d)i$, and define scalar multiplication as $c(a + bi) = ca + (cb)i$, where $c \in \mathbb{R}$. We check the eight axioms to show that these two operations define a vector space structure.

(i) The additive identity is $0+0i=0$ because for any $a+bi \in \mathbb{C}$, $(a+bi)+(0+0i)=(a+0)+(b+0)i=a+bi$ and $(0+0i)+(a+bi)=(0+a)+(0+b)i=a+bi$.
(ii) The additive inverse of $a+bi$ is $(-a)+(-b)i$ because $(a+bi)+((-a)+(-b)i)=(a+(-a))+(b+(-b))i=0+0i=((-a)+a))+((-b)+b)i=((-a)+(-b)i)+(a+bi)$.
(iii) Addition is commutative because $(a+bi)+(c+di)=(a+c)+(b+d)i=(c+a)+(d+b)i=(c+di)+(a+bi)$.
(iv) Addition is associative because $((a+bi)+(c+di))+(e+fi)=((a+c)+(b+d)i)+(e+fi)=(a+c+e)+(b+d+f)i=(a+bi)+((c+e)+(d+f)i)=(a+bi)+((c+di)+(e+fi))$.
(v) The multiplicative identity is 1: $1(a+bi)=(1\cdot a)+(1\cdot b)i=a+bi$.
(vi) The associative law for multiplication works: $c(d(a+bi))=c(da+(db)i)=(cda)+(cdb)i=(cd)(a+bi)$.
(vii) Scalar addition distributes: $(c+d)(a+bi)=((c+d)a)+((c+d)b)i=(ca+da)+(cb+db)i=(ca+(cb)i)+(da+(db)i)=c(a+bi)+d(a+bi)$.
(viii) Vector addition distributes: $e((a+bi)+(c+di))=e((a+c)+(b+d)i)=e(a+c)+(e(b+d))i=(ea+ec)+(eb+ed)i=(ea+(eb)i)+(ec+(ed)i)=e(a+bi)+e(c+di)$.

(b) This is true almost by definition of complex numbers, but we should be careful anyway. Any complex number is expressible as $a+bi$, where a and b are real. But $a+bi=(a+0i)+(0+bi)=(a(1)+(a\dot{0})i)+((b\dot{0})+(b\dot{1})i)=a(1+0i)+b(0+1i)$, so $a+bi$ is a real linear combination of $1=1+0i$ and $i=0+1i$. Hence 1 and i span $\mathbb{C}$.
Now suppose a and b are real and $a\dot{1}+b\dot{i}=0=0+0i$. Then $0+0i=a(1+0i)+b(0+1i)=(a+0i)+(0+bi)=a+bi$, which means that $a=b=0$ because a and b are real. (Because $0=a+bi \implies 0=(a+bi)(a-bi)=a^2+b^2$, and a^2 and b^2 are positive unless $a=b=0$.) Hence 1 and i are linearly independent.
By definition of a basis, 1 and i form a basis.

(c) We need to show that for any complex numbers z, z' and real numbers α and β, $A(\alpha z+\beta z')=\alpha A(z)+\beta A(z')$. Suppose $z=c+di$ and $z'=c'+d'i$. Then $A(\alpha z+\beta z')=A((\alpha c+\beta c')+(\alpha d+\beta d')i)=(a+bi)((\alpha c+\beta c')+(\alpha d+\beta d')i)=(a(\alpha c+\beta c')-b(\alpha d+\beta d'))+(a(\alpha d+\beta d')+b(\alpha c+\beta c'))i=\big((\alpha ac-\alpha bd)+(\alpha ad+\alpha bc)i\big)+\big((\beta ac'-\beta bd')+(\beta ad'+\beta bc')i\big)=\alpha((ac-bd)+(ad+bc)i)+\beta((ac'-bd')+(ad'+bc')i)=\alpha(a+bi)(c+di)+\beta(a+bi)(c'+d'i)=\alpha A(z)+\beta A(z')$.

(d) $A(1) = a + bi = (a)1 + (b)i$ and $A(i) = (a + bi)i = -b + ai = (-b)1 + (a)i$, so with respect to the basis $\{1, i\}$ the matrix is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

4. (Matt) No. One counterexample is $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$.

This problem could have been done just by plugging entries into A and B to find matrices for which $AB = 0$ but $BA \neq 0$. But we can gain more insight into the meaning of the statement $AB = 0$ by examining $\operatorname{Im} B$ and $\ker A$, where A and B are considered as linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. $AB = 0$ means that $\forall v \in \mathbb{R}^2, (AB)v = 0$ and thus $A(Bv) = 0$. So $\operatorname{Im} B \subset \ker A$.

We have three cases to consider, based on the dimension of $\operatorname{Im} B$.

$\dim(\operatorname{Im} B) = 2$: Then $\operatorname{Im} B = \mathbb{R}^2$, so $AB = 0 \implies A(Bv) = 0$ for all $v \in \mathbb{R}^2 \implies Aw = 0$ for all $w \in \mathbb{R}^2 \implies A = 0 \implies BA = 0$.

$\dim(\operatorname{Im} B) = 0$: The only subspace of dimension 0 is $\{0\}$. So $Bv = 0$ for all $v \in \mathbb{R}^2 \implies B = 0 \implies BA = 0$.

$\dim(\operatorname{Im} B) = 1$: This is the interesting case. Here we can choose any nonzero vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ to be a basis for Img B. We construct B such that the columns of B are multiplies of v: $B = \begin{bmatrix} av_1 & bv_1 \\ av_2 & bv_2 \end{bmatrix}$, where $a, b \in \mathbb{R}$. Then we can construct our matrix A such that $AB = 0$, or $\operatorname{Im} B \subset \ker A$. To do this we need the product of each row of A and each column of B to be zero. So we can write A in the following format:

$A = \begin{bmatrix} cv_2 & -cv_1 \\ dv_2 & -dv_1 \end{bmatrix}$, where $c, d \in \mathbb{R}$.

We have found a matrix such that $AB = 0$. BA, on the other hand, is $\begin{bmatrix} (ac + bd)v_1v_2 & -(ac + bd){v_1}^2 \\ (ac + bd){v_2}^2 & -(ac + bd)v_1v_2 \end{bmatrix}$. This matrix is nonzero as long as $ac + bd \neq 0$.

5. (Nils)

(a) First, recall that if $L \subsetneq M$ are (finite dimensional) vector spaces, then $\dim L < \dim M$, since you can pick a basis $\{v_1, \dots, v_l\}$ of L, and then add in $w \in M \backslash L$, so $\{v_1, \dots, v_l, w\} \subset M$ is linearly independent.

Thus, given

$$\{0\} \subset L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_r \subset V \tag{2}$$

we obtain

$$0 \leq \dim L_1 < \dim L_2 < \cdots < \dim L_r \leq d \tag{3}$$

so $r \leq d+1$, otherwise we'd have $\dim L_i = \dim L_{i+1}$ for some i.

(b) Consider $L_i = \ker A^i$. Since $A(0) = 0$, we see that if $v \in \ker A^i$, then $v \in \ker A^{i+1}$ (once you get to zero, you stay there). Now consider

$$\{0\} = L_0 \subset L_1 \subset L_2 \subset L_5 = V. \tag{4}$$

Since this is a sequence of subspaces of length $4 > 2+1$, by part (a), one of these containments must be an equality; we wish to show that it must be the last. If $L_0 = L_1$, then $\ker A = \{0\}$, so A is invertible and thus $A^5 \neq 0$, so this isn't the case.

Now suppose $L_1 = L_2$; then we claim that $L_i = L_{i+1}$ for $i > 1$. This is the crux of the problem, and is basically a generalization of the case where A is invertible. That is, consider $\operatorname{Im} A^i$. We claim that $\operatorname{Im} A^i \supset \operatorname{Im} A^{i+1}$; this is because $\operatorname{Im} A^{i+1}$ is the image of $\operatorname{Im} A$ under A^i, and thus is a subset of the image of V under A^i. Now by rank-nullity (the dimension theorem), we have that $\dim \operatorname{Im} A^1 = \dim A^2$; thus, $\operatorname{Im} A^1 = \operatorname{Im} A^2$, so if we *restrict* A to $\operatorname{Im} A$, it is fact an isomorphism on $\operatorname{Im} A$. Thus, $\operatorname{Im} A^1 = \operatorname{Im} A^2 = \operatorname{Im} A^3 = \cdots$, and thus $L_1 = L_2 = L_3 = \cdots$. Thus, if any of the containments is an equality (and one must be, by part (a)), then all further ones must be, so in particular $L_2 = L_5 = V$, so $A^2 = 0$.[1]

[1] I had this problem on my first midterm (generalized such that $2 \mapsto n, 5 \mapsto m > n$). It was a take-home. I got it wrong.