MATH 23A SOLUTION SET 11

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1. We have $f(x) = e^x$.

   If $f \in C^\infty(\mathbb{R}), a \in \mathbb{R}$ we define the Taylor series $P(a,x)$ for f at a by

$$P(a,x) = \sum_{m \geq 0} \frac{f^{(m)}(a)}{m!}(x-a)^m. \tag{1}$$

   $P(a,x)$ is the limit as m approaches ∞ of the mth degree Taylor polynomial

$$P_m(a,x) = \sum_{0 \leq i \leq m} \frac{f^{(i)}(a)}{i!}(x-a)^i. \tag{2}$$

   Define the remainder by $R_m(a,x) = f(x) - P_m(x)$.

   We know from calculus that f is continuous and differentiable with $f'(x) = f(x)$ for all x. Inductively, we can see that f is infinitely continuously differentiable. We need to show that for all $a, x \in \mathbb{R}$, $\lim_{m\to\infty} P_m(a,x) = e^x$. This is equivalent to showing that for all $a, x \in \mathbb{R}$, $\lim_{m\to\infty} R_m(a,x) = 0$.

   Let $a, x \in \mathbb{R}$. We showed in class that there is a number $c \in \mathbb{R}$ between x and a such that $R_m(a,x) = \frac{f^{(m+1)}(c)}{(m+1)!}(x-a)^{m+1} = \frac{e^c}{(m+1)!}(x-a)^{m+1}$.

   So $|R_m(a,x)| \leq \max_{c \in [a,x] \cup [x,a]} \frac{e^c}{(m+1)!}|(x-a)^{m+1}| = \frac{e^{\max_{a,x}}}{(m+1)!}|x-a|^{m+1}$.

   Let M be the smallest integer greater than or equal to $2|x-a|-1$. So $\frac{|x-a|}{M+1} \leq \frac{1}{2}$. Then $|R_M(a,x)| \leq \frac{e^{max_{a,x}}}{(M+1)!}|x-a|^{M+1}$ and for any $m > M$,

$$\begin{aligned}
|R_m(a,x)| &\leq \frac{e^{max_{a,x}}}{(M+1)!(M+2)\dots m}|x-a|^{M+1}|x-a|^{m-(M+1)} \\
&\leq \frac{e^{max_{a,x}}}{(M+1)!(M+1)^{m-(M+1)}}|x-a|^{M+1}|x-a|^{m-(M+1)} \\
&\leq \frac{e^{max_{a,x}}}{(M+1)!}|x-a|^{M+1}\left(\frac{|x-a|}{M+1}\right)^{m-(M+1)} \\
&\leq \frac{e^{max_{a,x}}}{(M+1)!}|x-a|^{M+1}\left(\frac{1}{2}\right)^{m-(M+1)} \\
&\leq \frac{e^{max_{a,x}}}{(M+1)!2^{(M+1)}}|x-a|^{M+1}\left(\frac{1}{2}\right)^{m}.
\end{aligned}$$

   Keep in mind that M depends only on a and m, which are fixed. So $|R_M(a,x)| \leq k\left(\frac{1}{2}\right)^m$, where k is a constant. So, clearly, $\lim_{m\to\infty}|R_M(a,x)| = 0$. So, for all $x \in \mathbb{R}$, $P(a,x) = \lim_{m\to\infty} P_m(a,x) = e^x$. QED
2. Let f be a function on $\mathbb{R}$ such that $f(x) = 0$ if $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$.

(a) First we will show that f is infinitely continuously differentiable as a function defined on the open interval $(0, \infty)$. For $x > 0$, $f(x) = e^{-1/x}$. We will show, inductively, that f is infinitely differentiable and for all integers $m \geq 0$ and $x > 0$, $f^{(m)}(x) = \frac{Q_m(x)}{x^{n_m}} e^{-1/x}$ for some polynomial $Q_m(x)$ and nonnegative integer n_m..
Base case: $f(x) = f^{(0)}(x) = \frac{Q_0(x)}{x^0} e^{-1/x}$, where $Q_0(x) = 1$.
Inductive step: Suppose $f^{(k-1)}(x) = \frac{Q_{k-1}(x)}{x^{n_{k-1}}} e^{-1/x}$, where $Q_{k-1}(x)$ is a polynomial and n_{k-1} is a nonnegative integer. By the rules for differentiation,

$$
\begin{aligned}
f^{(k)}(x) &= \frac{Q_{k-1}(x)}{x^{n_{k-1}}} \left(\frac{1}{x^2} e^{-1/x} \right) + e^{-1/x} \left(\frac{Q'_{k-1}(x) x^{n_{k-1}} - Q_{k-1}(x) n_{k-1} x^{n_{k-1}-1}}{x^{2(n_{k-1})}} \right) \\
&= \left(\frac{Q_{k-1}(x)}{x^{n_{k-1}+2}} + \frac{Q'_{k-1}(x) x^{n_{k-1}} - Q_{k-1}(x) n_{k-1} x^{n_{k-1}-1}}{x^{2(n_{k-1})}} \right) e^{-1/x} \\
&= \frac{Q_k(x)}{x^{n_k}} e^{-1/x}
\end{aligned}
$$

for some polynomial $Q_k(x)$ and nonnegative integer n_k.
So, for all integers $m \geq 0$ and all $x > 0$, $f^{(m)}(x) = \frac{Q_m(x)}{x^{n_m}} e^{-1/x}$ for some polynomial $Q_m(x)$ and nonnegative integer n_m. Further, for all $m \geq 0$, since $f^{(m+1)}$ exists, $f^{(m)}$ is differentiable and therefore continuous. So f is infinitely continuously differentiable on $(0, \infty)$.
Now we will show that f is infinitely continuously differentiable as a function defined on the open interval $(-\infty, 0)$. $f(x) = 0$ for all $x \in (-\infty, 0)$. We see that $f'(x) = f(x) = 0$ for all $x \in (-\infty, 0)$. It is easy to show by induction that $f^{(m)}(x) = 0$ for all integer $m \geq 0$ and all $x \in (-\infty, 0)$. So f is infinitely continuously differentiable on $(-\infty, 0)$.
Now we will show that, for all integer $m \geq 0$, $f^{(m)}(0) = 0$. We will show this inductively.
Base case: $f(0) = 0$ by definition.
Inductive step: Suppose $f^{(m-1)}(0) = 0$. Then

$$
\begin{aligned}
f^{(m)}(0) &= \lim_{h \to 0} \frac{f^{(m-1)}(0+h) - f^{(m-1)}(0)}{h} \\
&= \lim_{h \to 0} \frac{f^{(m-1)}(h)}{h}.
\end{aligned}
$$

We need to show that for every $\epsilon > 0$ we can find a $\delta > 0$ such that whenever $|h - 0| < \delta$, $\frac{f^{(m-1)}(h)}{h} < \epsilon$. For every $\epsilon > 0$ and $\delta > 0$, whenever $-\delta < h < 0$, $\frac{f^{(m-1)}(h)}{h} = 0 < \epsilon$. So we merely need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < h < \delta$, $\frac{f^{(m-1)}(h)}{h} < \epsilon$. So we need to show that $\lim_{h \to 0+} \frac{f^{(m-1)}(h)}{h} = 0$.
From our treatment of the derivatives of f on $(0, \infty)$, we know that $\lim_{h \to 0+} \frac{f^{(m-1)}(h)}{h} = \lim_{h \to 0+} \frac{Q_{m-1}(h)}{h^{n_{m-1}}} e^{-1/h}$ for some polynomial $Q_{m-1}(x)$ and nonnegative integer n_{m-1}. Let $t = 1/h$. Then this limit is equal to

$$
\lim_{t \to \infty} \frac{Q_{m-1}(1/t)}{(1/t)^{n_{m-1}}} e^{-t}
$$

$$= \lim_{t\to\infty} t^{n_{m-1}} Q_{m-1}(1/t)e^{-t}$$

$$= \lim_{t\to\infty} \frac{t^{n_{m-1}}A(t)}{t^c e^t}$$

expressing $Q_{m-1}(1/t)$ as the quotient of some polynomial $A(t)$ and t^c, where c is some nonnegative integer. This limit is clearly zero, as e^t grows faster than any polynomial or quotient of polynomials. So $f^{(m)}(0) = 0$.
So, for all integer $m \geq 0$, $f^{(m)}(0) = 0$. For any integer $m \geq 0$, $f^{(m+1)}(0)$ exists, so $f^{(m)}$ is differentiable at 0 and therefore continuous at 0.
So $f \in C^\infty(\mathbb{R})$.

(b) As we just showed, $f^m(0) = 0$ for all $m \geq 0$. So the Taylor series for f at a is $P(a,x) = \sum_{m\geq 0} \frac{f^{(m)}(a)}{m!}(x-a)^m = \sum_{m\geq 0} \frac{0}{m!}(x-a)^m = \sum_{m\geq 0} 0$, a sum which obviously converges to 0, regardless of x.

(c) Since all of the derivatives of f at 0 are 0, the Taylor series $P(0,x)$ for f at 0 converges to the constant function 0. But for $x > 0$, $f(x) > 0 = P(0,x)$. The reason for this nonconvergence is that having all derivatives equal to 0 at a point does not mean that a function is constant.

3. (a) Since V is a finite-dimensional vector space over $\mathbb{R}$, we know that for some n, V is isomorphic to $\mathbb{R}^n$ as an algebraic structure. We will assume that limits exist on V, so we can do calculus on V.
To avoid specifying v as a variable, we will assume that we are considering the Taylor polynomial at v, which is a function of x.
For this case we don't need the isomorphism. We define the linear Taylor polynomial $P_1(x) = f(v) + Df(v)(x-v)$. We easily verify that this is the Taylor polynomial as follows:
$P_1(x) = f(v) + Df(v)(x) - Df(v)(v)$. Since the first and third terms are constant and $Df(v)(x)$ is a linear function of x, $DP_1(v) = Df(v)$.
So $P_1(x)$ is the Taylor polynomial of f at v.

(b) A special case of below.

(c) Let V be a finite-dimensional vector space over $\mathbb{R}$, $f \in C^\infty(V)$, $v \in V$. A Taylor polynomial of degree k is a polynomial $P_k(x)$, not necessarily homogeneous, of degree $\leq k$ such that $\lim_{h\to 0} \frac{f(v+h)-P_k(v+h)}{|h|^k} = 0$. We will show that it suffices to find unique $Q_1, \ldots, Q_n$, where each Q_m is either a homogeneous polynomial of degree m or the zero polynomial, such that

$$\lim_{h\to 0} \frac{f(v+h) - f(v) - Q_1(h) - Q_2(h) - \cdots - Q_k(h)}{|h|^k} = 0. \tag{3}$$

Let $Q(h) = f(v) + Q_1(h) + \cdots + Q_k(h)$. Since is $Q_m(h)$ is either a homogeneous polynomial of degree m or the zero polynomial and $f(v)$ is a homogeneous polynomial of degree 0 in h, $Q(h)$ is a polynomial of degree $\leq m$. Since v is a constant, $P_k(x) = Q(x-v)$ is a polynomial of degree $\leq m$ which, by definition, satisfies $\lim_{h\to 0} \frac{f(v+h)-P_k(v+h)}{|h|^k} = 0$.
Choose a basis $\{e_1, \ldots, e_n\}$ of V. For $h = h_1e_1 + \cdots + h_ne_n$, let $Q_m(h) = \sum_{I\in I_n^m} \frac{1}{I!} D_I f(v) h^I$, where the $I = (i_1, \ldots, i_n)$ are multi-exponents with n entries and total degree m, $D_I f = D_1^{i_1} D_2^{i_2} \ldots D_n^{i_n} f$, $D_i f$ is the derivative with respect to e_i of f, and $h^I = h_1^{i_1} h_2^{i_2} \ldots h_n^{i_n}$. Since $i_1 + \cdots + i_n = m$,

$Q_i(h)$ is a homogeneous polynomial of degree m by the 2nd definition of homogeneous polynomial from class.
Considering V as isomorphic to $\mathbb{R}^n$ through the isomorphism that maps $\{e_1, \ldots, e_n\}$ to the elementary basis in $\mathbb{R}^n$, $P_k(x)$ is the polynomial in V described by Definition 3.3.15. Theorem 3.3.18 tells us that P_k is the unique polynomial of degree $\leq k$ in the basis $\{e_1, \ldots, e_n\}$ satisfying $\lim_{h\to 0} \frac{f(v+h)-P_k(v+h)}{|h|^k} = 0$.
Now comes the part that I forgot in my first solution set: showing that this Taylor polynomial is basis-independent. Repeat the above construction with another basis $\{d_1, \ldots, d_n\}$ to generate a new Taylor polynomial $S_k(x)$. Since each e_i can be expressed as a linear combination of $\{d_1, \ldots, d_n\}$, S_k is a polynomial of degree $\leq k$ in the basis $\{e_1, \ldots, e_n\}$ satisfying $\lim_{h\to 0} \frac{f(v+h)-P_k(v+h)}{|h|^k} = 0$. By the uniqueness of P_k, S_k is identically equal to P_k. So our definition of Taylor polynomial is basis-independent.