MATH 23A
SOLUTION SET 9

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1: (a) First, $f(2,2) = 0$, so $(2,2) \in C_0$. Second,

$$[Df](x,y) = \begin{bmatrix} 3x^2 + 6x - 12 & 2y - 6 \end{bmatrix}, \tag{9.1}$$

so $Df(2,2) = \left[\begin{smallmatrix} 12 & -2 \end{smallmatrix}\right]$, which is not equal to zero, so $(2,2)$ is a smooth point.

(b) Since $D_2f(2,2) \neq 0, D_1f(2,2) \neq 0$, we conclude from the inverse function theorem that (respectively), there exist ϵ_f, ϵ_g such that $y = f(x), x = g(y)$ on $(2 - \epsilon_f, 2 + \epsilon_f$ and $(2 - \epsilon_g, 2 + \epsilon_g)$, respectively. Taking $\epsilon = \min(\epsilon_f, \epsilon_g)$ yields the desired result.

(c) The tangent line of the curve $f(x,y) = 0$ at (x_0, y_0) is given by $[D_1f](x_0,y_0) \cdot (x - x_0) + [D_2f](x_0,y_0) \cdot (y - y_0) = 0$. In our case, this yields $12(x-2) - 2(y-2) = 0$.
The tangent line is a kind of an affine copy of the tangent space, which is given by $[D_1f](x_0,y_0)\dot{x} + [D_2f](x_0,y_0)\dot{y} = 0$, or in our case $12\dot{x} - 2\dot{y} = 0$.

(d) It is natural to solve this and the next part simultaneously. C_c will be singular at *some* point iff $Df = \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]$ at some point of C_c. Now $Df = 0$ for $(x,y) = (-1 \pm \sqrt{5}, 3)$ (by solving the linear and quadratic polynomials in (9.1)), so C_c is singular iff $(-1 \pm \sqrt{5}, 3) \in C_c$, i.e., iff $c = f(-1 \pm \sqrt{5}, 3)$.

(e) See above; the singular points of $C_{f(-1\pm\sqrt{5},3)}$ are $(-1 \pm \sqrt{5}, 3)$, respectively.

(f) By reasoning as above, $Df(x,y) = \left[\begin{smallmatrix} p'(x) & -q'(y) \end{smallmatrix}\right]$. This is zero iff $p'(x) = q'(y) = 0$. Now p', q' are polynomials of degrees $m-1, n-1$, respectively, so they have roots (possibly repeated) $x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}$. So (x,y) is a singular point iff $Df(x,y) = 0$, that is, for (x_i, y_j). Now these points lie on the curves $C_{f(x_i,y_j)}$, and thus there are at most $(m-1)(n-1) < mn$ singular curves.

2: (a) Note that squaring is locally C^1 invertible at a point iff the Jacobian at that point is invertible (and the partials are C^1, which they are). Via the attached Mathematica output, we see that

$$DS\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 2a & & & \\ & a+b & & \\ & & a+b & \\ & & & 2b \end{bmatrix}. \tag{9.2}$$

Further, note that a diagonal matrix is invertible iff all its diagonal entries are non-zero, and we see that the desired g exists iff $2a \neq 0$, $2b \neq 0$, and $a + b \neq 0$. A slick way of saying all of this is: $ab(a+b) \neq 0$.

(b) As above, but now we get

$$DS\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 5a^4 & & & \\ & a^4 + a^3b + a^2b^2 + ab^3 + b^4 & & \\ & & a^4 + a^3b + a^2b^2 + ab^3 + b^4 & \\ & & & 5b^4 \end{bmatrix}, \tag{9.3}$$

so g exists iff $ab(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \neq 0$.
Incidentally, you may have noticed that the big middle term is also $\frac{a^5 - b^5}{a-b}$, so the middle term factors as $\prod_{i=1}^{4}(a - \zeta_5^i b)$, where $\zeta = e^{2\pi i/5}$ (we call ζ_5 a *primitive fifth root of unity*). This is because $a^n - b^n = \prod_{i=1}^{n}(a - \zeta_n^i b)$, and we've just gotten rid of the $(a-b)$ term.
Looking at this, one could conjecture that for diagonal matrices, the map $A \mapsto A^n$ is locally C^1 invertible iff $ab\frac{a^n - b^n}{a-b} \neq 0$ (where one actually divides out the fraction). Can you prove this? Hint: it follows almost immediately from the expansion $Df(A)(H) = HA^{n-1} + AHA^{n-2} + \cdots + A^{n-1}H$ for $f(A) = A^n$.