MATH 23A SOLUTION SET 7

GEORGE KIRKUP

1.7.9) This problem is essentially trivial, so since everyone did quite well on it I will not say any more.

1.7.12a) The claim is that $L(M) = (\boldsymbol{D}f(A))M = A^2M + MAM + MA^2$. I will skip steps in this calculation.

(1)

$$\begin{aligned}
\lim_{H\to 0} \frac{|f(A+H) - f(A) - L(H)|}{|H|} &= \lim_{H\to 0} \frac{|H^2A + HAH + AH^2 + H^3|}{|H|} \\
&\le \lim_{H\to 0} \frac{|H^2A| + |HAH| + |AH^2| + |H^3|}{|H|} \\
&\le \lim_{H\to 0} \frac{|H|^2|A| + |H||A||H| + |A||H|^2 + |H|^3}{|H|} \\
&= \lim_{H\to 0} (|H||A| + |H||A| + |A||H| + |H|^2) \\
&= 0
\end{aligned}$$

Of course, the above limit is at least 0 so we are done.

1.7.16b) Evaluating the limit of f along the line $y = x$, we have

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x^2}$$

which we can easily see is $1/2$ using L'Hospital's Rule if nothing else. This disagrees with $f(0) = 0$ so the function is not continuous much less differentiable.

1.7.17d) This was also an exercise in partial derivatives.

1.7.21) Let $f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$ be a map from $\mathbb{R}^4$ to $\mathbb{R}$. Calculating the partial derivatives at $(1, 0, 0, 1)$ we see that

$$\boldsymbol{D}f(1,0,0,1) = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

so

$$(\boldsymbol{D}f(1,0,0,1)) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a + d.$$

Note that this is equivalent to the desired result.

1.8.3) Interestingly enough the easiest way to do this problem was by composing the functions to get

$$(f \circ \gamma)(t) = \sum_{1}^{n-1} t^{2i+1}$$

so it is clear that the derivative is

$$\sum_{1}^{n-1} (2i+1)t^{2i}.$$

1.8.4) Suppose there is a smooth map g with

$$g(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$(f \circ g)(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} y \\ x \end{pmatrix}.$$

By the chain rule we have

$$(\boldsymbol{D}f(\begin{pmatrix} 0 \\ 0 \end{pmatrix}))(\boldsymbol{D}g(\begin{pmatrix} 1 \\ 1 \end{pmatrix})) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

but

$$(\boldsymbol{D}f(\begin{pmatrix} 0 \\ 0 \end{pmatrix})) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so the above product is impossible. The easiest way to see this is that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is invertible while

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is not.