MATH 23A PROBLEM SET 5

due October 27th

Let $f\colon V \times V \to \mathbb{R}$ be a bilinear form. We say that f is symmetric if $f(v_1, v_2) = f(v_2, v_1) \forall v_1, v_2 \in V$. We say that f is anti-symmetric if $f(v_1, v_2) = -f(v_2, v_1) \forall v_1, v_2 \in V$.

1. Let V be a two-dimensional vector space, and $\{e_1, e_2\}$ and $\{v_1, v_2\}$ be two bases of V, such that $v_1 = e_1 + e_2, v_2 = e_1 - e_2$ and $f\colon V \times V \to \mathbb{R}$ be the bilinear form such that $f(v_1, v_1) = f(v_2, v_1) = f(v_2, v_2) = 1$ and $f(v_1, v_2) = 0$.
   (a) Find $f(e_1, e_2)$.
   (b) Show that there exists unique pair f_s, f_{as} of bilinear forms on V such that f_s is symmetric, f_{an} is anti-symmetric and $f = f_s + f_{an}$.
   (c) Find $f_s(e_1, e_2)$.
2. Let f be a bilinear form on a vector space V and $B = \{v_1, \ldots, v_d\} \in V$ be a basis of V. Consider the $d \times d$ matrix $M^B(f)$ such that $M^B(f)_{i,j} = f(v_i, v_j), 1 \leq i, j \leq d$.
   (a) Show that the bilinear form f is symmetric if and only if $M^B(f)_{i,j} = M^B(f)_{j,i} \forall i, j, 1 \leq i, j \leq d$ for some basis B in V. Show also that in this case $M^B(f)_{i,j} = M^B(f)_{j,i} \forall i, j, 1 \leq i, j \leq d$ for every basis B in V.
   (b) Show that the bilinear form f is antisymmetric if and only if $M^B(f)_{i,j} = -M^B(f)_{j,i} \forall i, j, 1 \leq i, j \leq d$ for some basis B in V.
   (c) Let $V = \mathbb{R}^3$, f be a bilinear form on V such that $f(e_i, e_j) = \delta^i_j$ where $\delta^i_j = 1$ if $i = j$ and 0 otherwise. Let $v_1 = e_1 + e_2 + e_3, v_2 = e_1 - e_2 + e_3, v_3 = e_1 + e_2 - e_3$. Show that $B = \{v_1, v_2, v_3\}$ is a basis of V and write down the matrix $M^B(f)$.
3. Let P_3 be the vector space of polynomials of degree ≤ 3 and $(,)\colon P_3 \times P_3 \to \mathbb{R}$ be the bilinear form given by $(p(x), q(x)) = \int_{x=-1}^{x=1} p(x)q(x)dx$.
   (a) Show that $P_3, (,)$ is an Euclidean space.
   Let $B = \{v_0, v_1, v_2, v_3\}$ be a basis in P_3 where $v_0 = 1, v_1 = x, v_2 = x^2, v_3 = x^3$.
   (b) Find real numbers a, b, c, d, e, f such that vectors $w_0 = v_0, w_1 = v_1 + av_0, w_2 = v_2 + bv_1 + cv_0, w_3 = v_3 + dv_2 + ev_1 + fv_0$ form an orthogonal basis of P_3.

(c) Find positive real numbers $r_i, 0 \leq i \leq 3$ such that vectors $e_i = r_i w_i, 0 \leq i \leq 3$ form an orthonormal basis of P_3.

(d) EXTRA CREDIT. Replace P_3 by P_n and show that $v_n = n!/(2n)!(d/dx)^n(x^2-1)^n$

4. Let $e_1, \ldots, e_d$ and $v_1, \ldots, v_d$ be two bases of a vector space V. Since $e_1, \ldots, e_d$ is a basis for any $j, 1 \leq j \leq d$ we can find numbers $a^i_j, 1 \leq i \leq d$ such that $v_j = \sum_{1 \leq i \leq d} a^i_j e_i$. Prove that the $d \times d$ matrix $A = \{a^i_j\}$ is invertible.
   Let f be a bilinear form on V such that $f(e_i, e_j) = \delta_{i,j}, 1 \leq i, j \leq d$ where $\delta_{i,j} = 1$ if $i = j$ and is equal to 0 otherwise. Find $f(v_i, v_j)$.