# MATH 23A SOLUTION SET 4

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1. Let $\{w_1, w_2, \ldots, w_l\} \in L$ be a basis for $L$, and $\{v_1, v_2, \ldots, v_m\}$ be a basis for $M$.

   Claim: $\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$ is a basis for $L \oplus M$.

   Proof: We will first prove that $\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$ spans $L \oplus M$. Any element $w$ of $L$ can be written as $c_1 w_1 + \cdots + c_l w_l$ for some $c_1, \ldots, c_l \in \mathbb{R}$. Any element $v$ of $M$ can be written as $d_1 v_1 + \cdots + d_m v_m$ for some $d_1, \ldots, d_m \in \mathbb{R}$. So any element $(w, v)$ of $L \oplus M$ can be written as

$$\begin{aligned}&(c_1 w_1 + \cdots + c_l w_l, d_1 v_1 + \cdots + d_m v_m)\\ &= c_1(w_1, 0) + \cdots + c_l(w_l, 0) + d_1(0, v_1) + \cdots + d_m(0, v_m)\end{aligned}$$

   for $c_1, \ldots, c_l, d_1, \ldots, d_m \in \mathbb{R}$. So $\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$ spans $L \oplus M$.

   Next we will prove that $\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$ is linearly independent. Suppose

$$c_1(w_1, 0) + \cdots + c_l(w_l, 0) + d_1(0, v_1) + \cdots + d_m(0, v_m) = (0, 0)$$

   for some $c_1, \ldots, c_l, d_1, \ldots, d_m \in \mathbb{R}$. Then

$$c_1 w_1 + \cdots + c_l w_l = 0.$$

   Since the $w_i$ are linearly independent,

$$c_1 = \cdots = c_l = 0$$

   Also,

$$d_1 v_1 + \cdots + d_m v_m = 0$$

   Since the $v_j$ are linearly independent,

$$d_1 = \cdots = d_m = 0$$

   Since

$$c_1 = \cdots = c_l = d_1 = \cdots = d_m = 0,$$

   $\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$ is linearly independent.

   Since it spans $L \oplus M$ and is linearly independent,

$$\{(w_1, 0), (w_2, 0), \ldots, (w_l, 0), (0, v_1), (0, v_2), \ldots, (0, v_m)\}$$

   is a basis for $L \oplus M$.

2. (a) Let $l_1, l_2 \in L, m_1, m_2 \in M, a \in \mathbb{R}$. Then

$$\begin{aligned}F((l_1, m_1) + (l_2, m_2)) &= F(l_1 + l_2, m_1 + m_2)\\ &= l_1 + l_2 + m_1 + m_2\\ &= F(l_1, m_1) + F(l_2, m_2).\end{aligned}$$

Also,

$$\begin{aligned} F(a(l_1, m_1)) &= F(al_1, am_1) \\ &= al_1 + am_1 \\ &= a(l_1 + m_1) \\ &= a(F(l_1, m_1)). \end{aligned}$$

(b) We need to show that $F$ is an isomorphism if and only if we can decompose any vector $v \in V$ into a unique sum $v = l + m$ for some $l \in L, m \in M$.
Suppose $F$ is an isomorphism. Let $v$ be an arbitrary element of $V$. Since $F$ is surjective, there exists $(l, m) \in L \oplus M$ such that $F(l, m) = v$. So $l + m = v$.
Now we will prove that this decomposition is unique. Suppose that for some $l_1, l_2 \in L$ and $m_1, m_2 \in M, l_1+m_1 = v = l_2+m_2$. Then $F(l_1, m_1) = v = F(l_2, m_2)$. Because $F$ is injective, we must have $(l_1, m_1) = (l_2, m_2)$, and so $l_1 = l_2$ and $m_1 = m_2$.
Conversely, suppose we can decompose any vector $v \in V$ into a unique sum $v = l+m$ for some $l \in L, m \in M$. Then there exists a unique ordered pair $(l, m), l \in L, m \in M$, such that $v = l + m$. Since $l + m = F(l, m)$ for all $l \in L$ and $m \in M$, there exists a unique ordered pair $(l, m) \in L \oplus M$ such that $F(l, m) = v$.

(c) Suppose that F is an isomorphism of vector spaces. Then ker $F = \{0\}$ and Img $F = V$. By the Rank-Nullity Theorem,

$$\dim V = 0 + \dim(L \oplus M).$$

So dim($V$)= $\dim(L \oplus M) = \dim(L) + \dim\ (M)$.
Suppose, for the sake of contradiction, that $L \cap M \neq \{0\}$. Then there exists $x$ such that $x \in L$ and $x \in M$. Since $M$ is a vector space, $-x \in M$. So $F(x, -x) = 0 = F(0, 0)$. So F is not one-to-one, so F is not an isomorphism. From this contradiction we can conclude that $L \cap M = \{0\}$.
Conversely, suppose $\dim(V) = \dim(L) + \dim\ (M)$ and $L \cap M = \{0\}$. Suppose $F(l_1, m_1) = F(l_2, m_2)$ for $l_1, l_2 \in L$ and $m_1, m_2 \in M$. Then $l_1+m_1 = l_2+m_2$, so $l_1-l_2 = m_2-m_1 \in L\cap M$. So $l_1-l_2 = 0 = m_1-m_2$, and therefore $l_1 = l_2$ and $m_1 = m_2$, so $(l_1, m_1) = (l_2, m_2)$ and $F$ is one-to-one.
Since $F$ is one-to-one, ker $F = \{0\}$ and dim(ker $F$) $= 0$. So, by the Rank-Nullity Theorem and using the result of Problem 1, dim(Img $F$) $= \dim(L \oplus M) = \dim(L) + \dim\ (M) = \dim(V)$. So any basis for Img $F$ is a set of $\dim(V)$ linearly independent vectors in $V$ and thus a basis for $V$. So Img $F = V$ and $F$ is onto. Since we know that $F$ is linear from part (a), $F$ is an isomorphism.

3. (a) Let $F, G \in V$ =Hom$(L, M)$ and $c \in \mathbb{R}$. Define $+\colon V \times V \to V$ by $(F + G)(w) = F(w) + G(w)$ for all $w \in W$. Define $\cdot\colon \mathbb{R} \times V \to V$ by $(cR)(w) = c(R(w))$ for all $w \in W$.
We will now verify that, with these operations as vector addition and scalar multiplication, Hom$(L, M)$ is a vector space. Hom$(L, M)$ is closed under vector addition and scalar multiplication by our definition of the domain and range of these functions. We then need to verify that Hom$(L, M)$ satisfies the eight axioms of a vector space given in Hubbard & Hubbard,

p. 189. I will verify the first one, and you should be able to verify the rest similarly.
(1) Additive identity. Define $Z \in \mathrm{Hom}(L, M)$ to be the function such that $Z(w) = 0$ for all $w \in W$. Then, for any $F$ in $\mathrm{Hom}(L, M)$, $(Z + F)(w) = Z(w) + F(w) = 0 + F(w) = F(w)$, so $Z + F = F$.

(b) To show that the map $A\colon \mathrm{Hom}(L, M) \to \mathrm{Mat}(l, m)$ is an isomorphism, we need to check that it is linear, that it is onto, and that it is one-to-one. Please note that $A$ is not what we normally think of as the matrix associated with a linear transformation. Each element $A(F)_{i,j}$ is the coefficient $c_j$ when we express $F(w_i)$ as the linear combination $c_1v_1 + c_2v_2 + \cdots + c_mv_m$. $A(F)$ is actually the transpose of the $m$-by-$l$ matrix we normally think of as associated with the linear transformation $F$.
Proof of linearity:

$$\begin{aligned} A(F+G)_{i,j} &= v^j((F+G)(w_i)) \\ &= v^j(F(w_i) + G(w_i) \\ &= v^j(F(w_i)) + v^j(G(w_i)) \\ &= A(F)_{ij} + A(G)_{i,j}. \end{aligned}$$

So $A(F + G) = A(F) + A(G)$.

$$\begin{aligned} A(cF)_{i,j} &= v^j((cF)w_i) \\ &= v^j(c(F(w_i))) \\ &= cv^j(F(w_i)) \\ &= c(A(F)_{i,j}). \end{aligned}$$

So $A(cF) = cA(F)$.
Proof that A is onto: Let $B \in \mathrm{Mat}(l, m)$. To define a linear map, it is sufficient to define its values on a basis. So we define $F\colon L \to M$ by $F(w_i) = \sum_{j=1}^{m} b_{i,j}v_j$. Then $A(F)_{ij} = v^j(F(w_i)) = v^j(\sum_{j=1}^{m} b_{i,j}v_j) = b_{i,j}$. So $A(F) = B$. So $F$ is onto.
Proof that A is one-to-one: It suffices to show that $\ker A = \{0\}$. Suppose that $F \in \ker A$. Then $A(F) = 0$. So, for all $1 \le i \le l$ and $1 \le j \le m$, $v^j(F(w_i)) = 0$. Since $v_1, \ldots, v_m$ is a basis for $M$, for all $i$ we can write $F(w_i) = c_{i,1}v_1 + \ldots c_{i,m}v_m$. Then, for all $i$ and $j$, $0 = v^j(F(w_i)) = c_{i,j}$. So, for all $i$, $F(w_i) = 0$. Since the value of $F$ at each basis vector is 0, $F(w) = 0$ for all $w \in W$ and $F = 0$. So $\ker A = \{0\}$, so $A$ is one-to-one. So A is an isomorphism.

(c) Using the Rank-Nullity Theorem on $A$,

$$\dim(\mathrm{Hom}(L, M)) = \dim(\ker A) + \dim(\mathrm{Img}\ A).$$

Since A is an isomorphism,

$$\dim(\mathrm{Hom}(L, M)) = 0 + \dim(\mathrm{Mat}(l, m)) = lm = \dim(L)\dim(M).$$

4. (a) For this part, we need to show that for any linear map $F\colon L \to M$ and linear functional $\lambda \in M^*$, and for any vectors $v_1, v_2 \in L$ and scalar $c \in \mathbb{R}$,

$$F^*(\lambda)(v_1 + v_2) = F^*(\lambda)(v_1) + F^*(\lambda)(v_2)$$

and
$$F^*(\lambda)(cv_1) = cF^*(\lambda)(v_1).$$
We have
$$\begin{aligned} F^*(\lambda)(v_1+v_2) &= \lambda(F(v_1+v_2)) \\ &= \lambda(F(v_1)+F(v_2)) \\ &= \lambda(F(v_1)) + \lambda(F(v_2)) \\ &= F^*(\lambda)(v_1) + F^*(\lambda)(v_2). \end{aligned}$$
Also,
$$\begin{aligned} F^*(\lambda)(cv_1) &= \lambda(F(cv_1)) \\ &= \lambda(cF(v_1)) \\ &= c\lambda(F(v_1)) \\ &= cF^*(\lambda)(v_1). \end{aligned}$$

(b) For this part, we need to show that for any linear map $F\colon L \to M$ and linear functionals $\lambda_1$ and $\lambda_2 \in M^*$, and for any scalar $c \in \mathbb{R}$,
$$F^*(\lambda_1+\lambda_2) = F^*(\lambda_1) + F^*(\lambda_2)$$
and
$$F^*(c\lambda_1) = cF^*(\lambda_1)$$
.
Note that we are comparing functions in $L^*$. To show that two functions in $L^*$ are equal, it suffices to show that at all $v \in L$, the two functions evaluate to the same value.
Let $v \in L$. Then
$$F^*(\lambda_1+\lambda_2)(v) = (\lambda_1+\lambda_2)(F(v)),$$
which by definition of addition in $M^*$ is equal to
$$\begin{aligned} &\lambda_1(F(v)) + \lambda_2(F(v)) \\ &= F^*(\lambda_1)(v) + F^*(\lambda_2)(v). \end{aligned}$$
So $F^*(\lambda_1+\lambda_2) = F^*(\lambda_1) + F^*(\lambda_2)$. Also,
$$\begin{aligned} F^*(c\lambda_1)(v) &= (c\lambda_1)(F(v)) \\ &= c\lambda_1(F(v)) \\ &= cF^*(\lambda_1)(v). \end{aligned}$$
So $F^*(c\lambda_1) = cF^*(\lambda_1)$. So the map $F^*\colon M^* \to L^*$ is linear.

(c) The solution to this problem is very short, but many of you who attended my office hours were confused with by the notation. One must be careful to apply the definition of $A$ very carefully in writing out $A(F^*)$. $A(F)_{i,j}$ is defined to be $v^j(F(w_i))$, where $v^j$ is the $j$-th vector in the dual basis of the range, and $w_i$ is the $i$-th vector in the basis of the domain. Since $F^*\colon M^* \to L^*$, the $j$-th vector in the dual basis of the range of $F^*$ is actually the basis vector dual to $w^j$. This turns out to be the element of $L^{**}$ representing evaluation at $w_j$. And the $i$-th vector in the basis of the domain is the element $v^i$ of $M^*$. So we have $A(F^*)_{i,j} =$ the evaluation at $w_j$ of $F^*(v^i) = F^*(v^i)(w_j) = v^i(F(w_j)) = A(F)_{j,i}$. So $A(F^*) = A^T(F)$.

5. (a) To show that $\lambda_1, \lambda_2, \lambda_3$ form a basis for $P_2^*$, it suffices to show that these three vectors are linearly independent. This is because $\dim(P_2^*)$=$\dim(P_2)$=3, and any three vectors in a space of dimension three are linearly independent.
Suppose
$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = 0$$
for $a_1 = a_2 = a_3 \in \mathbb{R}$. Then, for all $p(x) = c_0 + c_1x + c_2x^2$, we must have
$$a_1\lambda_1(p(x)) + a_2\lambda_2(p(x)) + a_3\lambda_3(p(x)) = 0.$$
By the definition of the $\lambda_i$,
$$a_1p(1) + a_2p(2) + a_3p(3) = 0.$$
So
$$a_1(c_0 + c_1 + c_2) + a_2(c_0 + 2c_1 + 4c_2) + a_3(c_0 + 3c_1 + 9c_2) = 0.$$
So
$$(a_1 + a_2 + a_3)c_0 + (a_1 + 2a_2 + 3a_3)c_1 + (a_1 + 4a_2 + 9a_3)c_3 = 0.$$
Since this equation has to hold for all values of the $c_i$, we must have
$$a_1 + a_2 + a_3 = 0,$$
$$a_1 + 2a_2 + 3a_3 = 0,$$
$$a_1 + 4a_2 + 9a_3 = 0.$$
The only solutions to this system of equations are $a_1 = a_2 = a_3 = 0$. So $\lambda_1, \lambda_2, \lambda_3$ are linearly independent and hence a basis for $P_2^*$.

(b) We are now asked to find the dual basis $\lambda_i^*, 1 \leq i \leq 3$, of the space $(P_2^*)^* = P_2$.
First, a review of fundamental concepts:
An element $p$ of $P_2$ is a polynomial in $x$ of degree $\leq 2$.
An element $\lambda$ of $P_2^*$ is a linear functional on $P_2$. We showed above that all elements of $P_2^*$ can be written as a linear combination of "evaluation at 1," "evaluation at 2," and "evaluation at 3."
An element $\lambda^*$ of $P_2^{**}$ is a linear functional on $P_2^*$. Any element of $P_2^{**}$ can be expressed as evaluation at a polynomial member of $P_2$, and in this way we can identify $P_2^{**}$ with $P_2$.
Now the search is on:
The dual basis $\lambda_i^*$ of $\lambda_i$ must satisfy the condition $\lambda_i^*(\lambda_j) = 1$ for $i = j$ and $\lambda_i^*(\lambda_j) = 0$ for $i \neq j$.
We will now look for polynomials $p_1$, $p_2$, and $p_3$ such that evaluation at $p_i$ satisfies the condition for $\lambda_i^*$. So we need $\lambda_j(p_i) = 1$ for $i = j$ and $\lambda_j(p_i) = 0$ for $i \neq j$.
Notice that these are exactly the same equations that we would have if we started out trying to solve for the basis $p_1, p_2, p_3$ of $P_2$ that the $\lambda_i$ are the dual of. In general, the dual of the dual of a basis consists of the linear functionals representing evaluation at the original basis vectors.
I will solve for $p_1$, and the other solutions are similar. We need a polynomial of degree $\leq 2$ such that $p_1(1) = 1, p_1(2) = 0, p_1(3) = 0$. One could write out these equations in terms of the coefficients of $p_1$ and solve three equations in three unknowns. Or simply note that since 2 and 3 are roots of $p_1$, $x - 2$ and $x - 3$ must be factors of $p_1$. And to get $p_1(1) = 1$, we

let $p_1 = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}x^2 - \frac{5}{2}x + 3$. Similarly, $p_2 = -x^2 + 4x - 3$ and $p_3 = \frac{1}{2}x^2 - \frac{3}{2}x + 1$. And each $\lambda_i^*$ is the function representing evaluation at the respective $p_i$.