MATH 23A PROBLEM SET 4

due October 22th

1. Let L, M be finite-dimensional vector spaces. Show that $\dim(L \oplus M) = \dim(L) + \dim(M)$.
2. Let V be a finite-dimensional vector space, $L, M \subset V$ subspaces of V. Consider the map $F\colon L \oplus M \to V$ given by $F(l, m) = l + m, l \in L, m \in M$.
   (a) Check that the map F is linear.
   (b) Show that F is an isomorphism of vector spaces [that is F is one-to-one and onto] if and only if we can decompose any vector $v \in V$ as a sum $v = l + m, l \in L, m \in M$ and such a decomposition is unique.
   (c) Show that F is an isomorphism of vector spaces if and only if $\dim(V) = \dim(L) + \dim(M)$ and $L \cap M = \{0\}$.
3. Let L, M be finite-dimensional vector spaces and $\mathrm{Hom}(L, M)$ be the set of linear maps F from L to M.
   (a) Define the structure of a vector space on the set $V = \mathrm{Hom}(L, M)$, that is, define two operations $+\colon V \times V \to V$, $\cdot\colon \mathbb{R} \times V \to V$ and show that with these as vector addition and scalar multiplication, $\mathrm{Hom}(L, M)$ is a vector space.
   (b) Let $\{w_1, \ldots, w_l\} \in L, \{v_1, \ldots, v_m\} \in M$ be bases of L and M, and $\left\{w^1, \ldots, w^l\right\}, \{v^1, \ldots, v^m\}$ be the dual bases of L^* and M^*. To any linear map F from L to M we associate an $l \times m$ matrix $A(F)$ such that $A(F)_{i,j} = v^j(F(w_i)), 1 \leq i \leq l, 1 \leq j \leq m$. Show that the map $\mathrm{Hom}(L, M) \to \mathrm{Mat}(l, m)$ is an isomorphism of vector spaces.
   (c) Prove that $\dim(\mathrm{Hom}(L, M)) = \dim(L)\dim(M)$.
4. Let L, M be vector spaces. To any linear map $F\colon L \to M$ and a linear functional $\lambda \in M^*$ we associate a function $F^*(\lambda) = F^*_\lambda\colon L \to \mathbb{R}$ by
$$F^*(\lambda)(v) = \lambda(F(v)), \forall v \in L$$

This can be summarized in the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "F"] \arrow[dr, dashed, "F^*_\lambda"'] & M \arrow[d, "\lambda"] \\ & \mathbb{R} \end{tikzcd}$$

Here's how to read a commutative diagram: it's just a summary of all the maps you have sitting around, with solid arrows for the maps you start with and dashed arrows for the maps you get from the existing maps. It is called "commutative" because it doesn't matter which path you take from one point to another; it's the same map. Thus, $F^*_\lambda = \lambda \circ F$, since they both start at L and end in $\mathbb{R}$.

Nils finds these diagrams to be really pretty and much easier to understand than formulas.

(a) Show that the function $F^*_\lambda \colon L \to \mathbb{R}$ is linear.

(b) Thus, F^* defines a map $F^* \colon M^* \to L^*$. Show that this map is linear.

(c) Let $\{w_1, \ldots, w_l\} \in L, \{v_1, \ldots, v_m\} \in M$ be bases of L and M, and $\left\{w^1, \ldots, w^l\right\}, \{v^1, \ldots, v^m\}$ be the dual bases of L^* and M^*. Show that for any linear map $F \colon L \to M$ the matrix $A(F^*)$ is equal to the transpose of the matrix $A(F)$.

5. Let P_2 be the linear space of polynomials $p(x)$ of degree ≤ 2. Consider the linear functionals $\lambda_i, 1 \leq i \leq 3$ on P_2 given by $\lambda_i(p(x)) = p(i)$.

(a) Show that $\lambda_i, 1 \leq i \leq 3$ is a basis of the space P_2^*.

(b) Find the dual basis $\lambda_i^*, 1 \leq i \leq 3$ of the space $(P_2^*)^* = P_2$.