MATH 23A
SOLUTION SET 1

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1.1.4: We begin with a lemma:

1.1. Lemma. *If V is a subspace of $\mathbf{R}^n$, then $\vec{0} \in V$.*

Note that the book defines a subspace (1.1.5, p. 33) to be a *non-empty* subset that is closed under vector addition and scalar multiplication. This is so that the empty set isn't considered a subspace; it is common in math to craft definitions such that certain degenerate cases are not allowed.

Proof. Since V is non-empty, pick $v \in V$. Then $\vec{0} = 0v \in V$, since it is closed under scalar multiplication. □

(a) Not a subspace: $0 = -2 \cdot 0 - 5$ yields $0 = -5$, so $\vec{0}$ is not in the line.
(b) Not a subspace: $0 = 2 \cdot 0 + 1$ yields $0 = 1$, so $\vec{0}$ is not in the line.
(c) This is a subspace of $\mathbf{R}^2$, or in fact $\mathbf{R}^n$. If $(x_1, y_1), (x_2, y_2)$ satisfy $y_i = \frac{5}{2}x_i$, then summing these yields $y_1 + y_2 = \frac{5}{2}(x_1 + x_2)$, and similarly for scalar multiplication.

The lines in (a) and (b) are called AFFINE SUBSETS, in that they look like vector subspaces, but translated over such that they don't contain $\vec{0}$.

1.1.5: See attached Mathematica output. Note that even Mathematica has trouble with 3-dimensional vector fields.

1.2.2:
(a) $\begin{bmatrix} 28 & 14 \\ 79 & 44 \end{bmatrix}$
(b) not possible
(c) $\begin{bmatrix} 3 & 0 & -5 \\ 4 & -1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 31 \\ -5 \\ -2 \end{bmatrix}$
(e) $\begin{bmatrix} -10 & 29 \\ -9 & 24 \end{bmatrix}$
(f) not possible

1.2.4:
(a) The third column of AB equals

$$AB \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \tag{1.2}$$

(b) The second row of AB equals

$$\begin{bmatrix} 0 & 1 \end{bmatrix} AB = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 16 & 2 \end{bmatrix}. \tag{1.3}$$

1.2.5:

$$AB = \begin{bmatrix} 1+a & 1 \\ 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 1 & 1 \\ 1+a & a \end{bmatrix} \tag{1.4}$$

Setting these equal shows that the commute iff

$$\begin{aligned} 1 + a &= 1 & \qquad 1 &= 1 \\ 1 &= 1 + a & \qquad 0 &= a \end{aligned}$$

This set of linear equations has the unique solution $a = 0$, so A, B commute iff $a = 0$.

1.2.12: Yup, it works.

Note that just because this formula works when $ad - bc \neq 0$ does *not* mean that a matrix is only invertible when $ad - bc \neq 0$; for all you know, another formula might work.

For a trivial example of this principle, note that the answer to $0 \cdot x$ is $0/x = 0$ when $x \neq 0$, and is still 0 when $x = 0$, but of course this formula doesn't work.

For a somewhat more interesting example, for what values of x does $ax = 0$? Well, if $a \neq 0$ then $0/a = 0$ is the only solution, while if $a = 0$ this formula doesn't work—but then *any* value of x is a solution. So the failure of a formula does not imply that a solution doesn't exist.

1.2.13: This was the most interesting problem on the set, the one that gave people the most trouble, and the one that admits the most solutions. I'll thus demonstrate several solutions.

- If you feel lost when faced with a problem, a good bet is always to try a few examples to get your bearings (I won't illustrate that here). Another good bet is to try a direct computation:
  Er, well, if A *does* have an inverse, then it must be of the form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.5}$$

  so the following equations must have a solution:

$$aw + by = 1 \qquad ax + bz = 0 \tag{1.6}$$
$$cw + dy = 0 \qquad cx + dz = 1 \tag{1.7}$$

  So now you want to solve this, so you assume $c \neq 0$ (say), so $w = -dy/c$, which you can plug in to get $-ady/c + by = 1$ so $(bc - ad)y = 1$ so $0 = 1$, which is a contradiction, so $c = 0$. By symmetry of the equations you can repeat this for a, b, d, and thus these all must be zero. But then $0x + 0z = 1$, which is a contradiction, so no such inverse exists.
  Proofs of this form, namely direct computations that reduce to case analysis, are often unpleasant to write and to read. However, these are sometimes the easiest or only way that you can solve a problem, so you should be comfortable with them. It is also sometimes the case that one must do a case analysis because the cases really are essentially different, though then you won't say: "And if we just replace a with c everywhere the same kinda thing works."
- If you're lucky, you may notice a clever trick that allows you to solve a problem[1]. For instance, in (1.6), you can multiply the top left and bottom right, and then top right and bottom left to obtain:

$$(aw + by)(cx + dz) = acwx + adwz + bcxy + bdyz = 1 \tag{1.8}$$
$$(ax + bz)(cw + dy) = acwx + adxy + bcwz + bdyz = 0. \tag{1.9}$$

  Now comparing these two equations and recalling that $ad = bc$, we see that (1.8) is equivalent (by switching ad, bc) to:

$$acwx + adxy + bcwz + bdyz = 1, \tag{1.10}$$

  which contradicts (1.9), so an inverse doesn't exist.
  Note that in mounting a frontal attack on a problem and writing down equations, you can sometimes notice a trick from an unexpected similarity in the expressions, and save yourself a lot of work.
- My favorite method of proof is to get a (geometric) understanding of what's going on, believe the result, and then see how to prove it from there. Sometimes this doesn't work, but it's good to try.
  Note that you'd only have come up with this particular solution if you were already comfortable with linear algebra, and we certainly didn't expect anyone to give this one. It actually encapsulates the algebraic case analysis of the first solution.
  If $ad - bc = 0$, then $ad = bc$, so $a/c = b/d$ (assuming that $cd \neq 0$ so that this is well-defined). So $A(e_1)$ and $A(e_2)$ (where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$) are both vectors with the same slope, i.e., on the same line. So A take $\mathbf{R}^2$ and squishes it down to a line, that is, it drops the dimension down, and thus isn't

[1] I personally find these unsatisfying, in that they can tell you that a result is true, but not why. I feel that the point of math is understanding, not proof. Proof serves as a check on your intuition, but not as a replacement. For someone who's thought a lot more about this than I, see Michael Atiyah's "Collected Works". However, it's often useful to be able to prove something without really understanding why.

invertible. To make this rigorous, we will exhibit a vector v that is sent to 0 by A. Since $A(0) = 0$ for any matrix, this will show that A is not injective (one-to-one), and thus doesn't have an inverse. Note that if two vectors v, w generate the same subspace, one must be a multiple of the other, say $w = kv$, so $w - kv = 0$, and they are linearly dependent. Thus, the above statement that $a/c = b/d$ indicates that one column is a multiple of the other; in particular the second column is b/a (or d/c) times the first one. So $(b/a, -1)$ and $(d/c, -1)$ both map to zero. Since we don't know that a, c are non-zero, we multiply through at get that $(b, -a)$ and $(d, -c)$ both map to zero—and either one of these vectors is non-zero (and thus A is not injective), or the whole matrix A is identically zero[2], in which case it is clearly not injective, and thus not invertible.

1.2.14: Often the easiest way to prove a result about matrices (or in fact any complex object, like sequences or sets), is to show it for an arbitrary element, and thus it holds for any element.

First note that the comparison of $(AB)^T$ and B^TA^T makes sense, in that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then both sides are $p \times m$ matrices.

Comparing terms, we obtain:

$$(AB)^T{}_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} \tag{1.11}$$

$$\left(B^TA^T\right)_{ij} = \sum_{k=1}^{n} B^T{}_{ik}A^T{}_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk} \tag{1.12}$$

which are equal, so the matrices are equal.

1.2.20:

(a) Yup, it works.

(b) If a map has a left and a right inverse, then they are equal. Thus, it suffices to check $\left[\begin{smallmatrix} a & 1 & 0 \\ b & 0 & 1 \end{smallmatrix}\right]$:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \neq I_3. \tag{1.13}$$

Thus, it has no right inverse.

(c) By 1.2.20a, $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ has infinitely many left inverses. By 1.2.14, a left inverse for B is a right inverse for B^T. Thus, B^T has infinitely many right inverses.

1.2.21: Okay, since the Hubbards are so confident that there is an upper-triangular inverse, let's assume there is and find it by a direct computation:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & y+az+b \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.14}$$

For this last to be the identity, we need $a + x = c + z = y + az + b = 0$, i.e., $x = -a, z = -c, y = b - ac$. Thus, $\left[\begin{smallmatrix} 1 & -a & b-ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{smallmatrix}\right]$ works. Note that a square matrix has a left inverse iff it has a right inverse, and thus this is the (unique) inverse.

[2] We say that a function is IDENTICALLY ZERO if it maps a whole set to zero. This is to distinguish it from mapping a particular point to zero. So for instance $x^2 - 4$ is zero at $x = \pm 2$, but it is not identically zero (the only polynomial that is being the zero polynomial, 0).