Mathematics 22a Homework Problems 5

The following problems are due on **Monday, October 26**. 1. 2.22

2. Suppose we have a two-particle system, with particles "A" and "B". Each particle can be in two different states, "up" or "down," and so there are four possible states for the system:

1. A up, B up
2. A up, B down
3. A down, B up
4. A down, B down

In each of the following cases, compute the 4×4 stochastic matrix P generating the Markov process of this system. Then compute Pv, where v is the column vector representing the probabilities:

Prob(state 1) $= .2$
Prob(state 2) $= .2$
Prob(state 3) $= .3$
Prob(state 4) $= .3$

a) The system obeys the following rules: If, in a given time interval, one particle is "up", then the probability of the *other* particle being "up" in the *next* time interval is .8. If one of the particles is "down" in a given time interval, then the probability of the other particle being down in the next time interval is .6.

b) The system obeys the following rules: If both particles have the same orientation in a given time interval, then with probability .7, they will have the same orientation in the next time interval and with equal probability one of the other possible outcomes will occur in the next time interval. If, on the other hand, both particles have different orientations in a given time interval, then with probability .6 they will both point "up" in the next time interval, and with probability .4 they will both point "down" in the next time interval.

3. Let $A = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$. Compute $\exp(A)$.

Hint: This is similar to computing $\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$. Recall (or derive, or look up) the power series for the hyperbolic trigonometric functions $\sinh(t)$ and $\cosh(t)$.

4. 3.4.

5. Prove that if A is a 2×2 matrix, then

$$\det(\exp(A)) = e^{\mathrm{tr}(A)}$$

as follows:

a) Show that for any 2×2 matrices B and C, $\det(BCB^{-1}) = \det(C)$ and $\mathrm{tr}(BCB^{-1}) = \mathrm{tr}(C)$.

b) Show that the claim is true for matrices of the form:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

c) Show the claim is true for any 2×2 matrix.

6. Show that is is *never* true that:

$$\exp\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^a & e^b \\ e^c & e^d \end{pmatrix}.$$

Hint: Use problem 5.

7. 3.11.