Mathematics 22a Homework Problems 10

The following problems are due on **Wednesday, December 9**.

1. Let f and ϕ be defined as below. In each case, calculate i) the pullback function $\phi^* f$, ii) the differential df and iii) the pulled-back differential $\phi^* df$.

a) $f : \mathbf{R}^3 \to \mathbf{R}$ given by the formula $f(x, y, z) = x^2 + y^2 - z^2$ and $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ given by $\phi(\rho, \varphi, \theta) = (\rho \cos\theta \sin\varphi, \rho\cos\theta\cos\varphi, \rho\sin\theta)$ ("spherical cooordinates").

b) $f : \mathbf{R}^3 \to \mathbf{R}$ given by the formula $f(x, y, z) = x^2 + y^2 - z^2$ and $\phi : \mathbf{R} \to \mathbf{R}^3$ given by $\phi(t) = (\cos(t), \sin(t), t)$. (The graph of ϕ is a helix.)

2. The **vector** or **cross product** of two vectors $\mathbf{v} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3)$ and $\mathbf{w} = (\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3)$ is defined by:

$$\mathbf{v} \times \mathbf{w} = (\mathbf{v}_2\mathbf{w}_3 - \mathbf{w}_2\mathbf{v}_3 \quad \mathbf{w}_1\mathbf{v}_3 - \mathbf{v}_1\mathbf{w}_3 \quad \mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2).$$

If you are familiar with the definition of the determinant of a 3×3 matrix, the following "definition" is easy to remember:

$$\mathbf{v} \times \mathbf{w} = \mathrm{Det}\begin{pmatrix} dx & \mathbf{v}_1 & \mathbf{w}_1 \\ dy & \mathbf{v}_2 & \mathbf{w}_2 \\ dz & \mathbf{v}_3 & \mathbf{w}_3 \end{pmatrix},$$

where dx dy and dz represent the row vectors $(1 \quad 0 \quad 0)$, $(0 \quad 1 \quad 0)$, and $(0 \quad 0 \quad 1)$, respectively. (To save space, I've written everything as row vectors. Just switch all rows to columns to get the cross product of two column vectors.)

The cross product has the following properties. The tedious proofs are left to the writers of textbooks and the student who has free time to kill.

1. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ and $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.
3. $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, for all real numbers a.
4. $< \mathbf{v}, \mathbf{v} \times \mathbf{w} > = < \mathbf{w}, \mathbf{v} \times \mathbf{w} > = 0$, where $< , >$ is the Euclidean scalar product (dot product) on $\mathbf{R}^3$.
5. $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Property 4 says that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

a) Prove that if $\mathbf{v}$ and $\mathbf{w}$ span a plane in $\mathbf{R}^3$ then $\mathbf{v} \times \mathbf{w}$ is orthogonal to that plane. (Note that if $\mathbf{u}$ is a vector in the plane spanned by $\mathbf{v}$ and $\mathbf{w}$ then $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, for some real numbers a and b).

b) Suppose that the plane P through the origin in $\mathbf{R}^3$ is spanned by the vectors $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$. Find the equation for P.

c) Let $p = (x_0, y_0, z_0)$ and let $P' = P + p$; that is, let P' be the translate of the plane P of problem 1 a) so that the origin is moved to the point p. Find an equation for P'.

d) A quote from the end Chapter 5 of *B&S*: "You should know how to determine the equation of the line or plane tangent to the graph of a function at a given point." Assuming you are familiar with finding the equation of a tangent line, let's see how to find the equation of a tangent plane to a surface.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function that is differentiable at the point (x_0, y_0). The **graph** of f is a surface in $\mathbf{R}^3$ and consists of the set of points:

$$\{(x, y, f(x, y)) \mid x, y \in \mathbf{R}\}.$$

Suppose that $\gamma : \mathbf{R} \to \mathbf{R}^2$ is a curve in the x, y-plane with $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma(0) = (x_0, y_0)$. **Find** the equation of the curve $\tilde{\gamma} : \mathbf{R} \to \mathbf{R}^3$ such that the image of $\tilde{\gamma}$ lies in the graph of f, $\tilde{\gamma}$ projects to γ in the x, y-plane, and $\tilde{\gamma}(0) = (x_0, y_0, f(x_0, y_0))$ (as in the illustration below).

e) Find the tangent vector $\dot{\tilde{\gamma}}(0)$ to the curve $\tilde{\gamma}$ at the point (x_0, y_0, z_0) (where $z_0 = f(x_0, y_0)$). The **tangent plane** to the graph of f at the point (x_0, y_0, z_0) is the set of all such tangent vectors. (All these vectors are based at the point (x_0, y_0, z_0), which is then considered to be the origin of the tangent plane).

f) Let α and β be defined by the equations $\alpha(t) = (x_0 + t, y_0)$ and $\beta(t) = (x_0, y_0 + t)$. Find the tangent vectors to the graph of f determined by these curves.

g) The tangent vectors in 1 f) span the tangent plane through (x_0, y_0, z_0). Find the equation of this plane.

3. Find the equation of the tangent plane to the surface $z = x^3 - 3xy^2$ at the point $(-1, 1, 2)$.

4. Find the angle between the surfaces $z = x^3 - 3xy^2$ and $z = x^2 + y^2$ at the point $(-1, 1, 2)$.

5. Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be given by: $\phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Find the pullback $\phi^*\omega$ of the differential one-form ω, where:

a) $\omega = 2x dx - 3y dy$.

b) $\omega = 2xy dx + x^2 dy - y^2 dy$.

c) $\omega = df$, where $f(x, y) = 2xy(2 - x^2)$.

6. If ω is a differential one-form on $\mathbf{R}$, then $\omega = g\,dx$, where $g : \mathbf{R} \to \mathbf{R}$. Does there exist an $f : \mathbf{R} \to \mathbf{R}$ such that $\omega = df$? Is there an example of a one-form $\omega = g_1 dx_1 + g_2 dx_2$ on $\mathbf{R}^2$ such that $\omega \neq df$, for any $f : \mathbf{R}^2 \to \mathbf{R}^2$?

7. B & S p. 218 # 5.9.