L-FUNCTIONS LECTURE 1

Outline:

1. Restricted direct product (rdp) (ANT p.322; section 3.1 of thesis).
2. Adeles $\mathbb{A}_K$. (called Valuation vectors V in thesis).
3. K is discrete in $\mathbb{A}_K$. The quotient $\mathbb{A}_K/K$ is compact.
4. Ideles $J_K = \mathbb{A}_K^*$, but the rdp topology in J is stronger than that induced from $\mathbb{A}$.
5. Statements of Pontrjagin duality for locally compact groups.: $G'' = G$, G compact $\Leftrightarrow$ G' discrete. Examples: $\mathbb{R}, \mathbb{Q}_p$ are self-dual. $\mathbb{Z}' = \mathbb{R}/\mathbb{Z}$, $\quad \mathbb{Q}' =$ solenoid.
6. Character group of an rdp is an rdp (ANT, p325,Th. 3.2.1, but $H^\perp$ is denoted by H^* in the thesis.
7. The character $\lambda_{\mathbb{Q}} = (\lambda_\infty, \lambda_2, \lambda_3, ..., \lambda_p, ...)$ of $\mathbb{A}_Q$.
8. $\mathbb{A}_Q$ is self-dual via $< x, y >= \lambda(xy)$. Also, $\mathbb{Q}^\perp = \mathbb{Q}$.

L-FUNCTIONS LECTURE 2

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1. Functorality of duality.

All groups are locally compact abelian (lca) and homomorphisms between them are continuous unless otherwise specified. A homomorphism $f : H \to G$ induces a dual, or adjoint, homomorphism $f' : G' \to H'$, defined by $< x, f'(y) >:=< f(x), y >$, for $x \in H, y \in G'$. Thus $G \mapsto G'$ is a contravariant functor giving an equivalence between the category of lca groups and its opposite (Pontrjagin duality theorem). Note that things are not too nice with general homomorphisms. For example, f is injective if and only if $f'(G')$ is dense in H', whereas one would like the condition to be that f' is surjective. Kernels exist, but not cokernels. However, if we define f to be a"strict homomorphism" if f induces an isomorphism of lca groups $H/\operatorname{Ker}(f) \to f(H)$ with $f(H)$ having the topology which it gets as subgroup of G, then the category of lca groups and *strict* homomorphisms is an abelian category. More naively: if H is a closed subgroup of G, then $(G/H)' = H^\perp$ where $H^\perp$ is the subgroup of G' consisting of the characters of G which are trivial on H, the inclusion map $H^\perp \hookrightarrow G'$ being the adjoint of the canonical projection $G \to G/H$. Of course $(H^\perp)^\perp = H$. Also, $(f')' = f$.

A simple example of a non-strict homomorphism is the inclusion map $\mathbb{Q} \hookrightarrow \mathbb{R}$, giving $\mathbb{Q}$ the discrete topology and $\mathbb{R}$ its usual topology. Even more horrible is the identity map from $\mathbb{R}$ with the discrete tpopology to $\mathbb{R}$ with the usual topology.

2. Examples of duality.

Abelian profinite groups are the duals of discrete abelian torsion groups. The former are inverse limits and the latter direct limits, of finite abelian groups. Special cases: $\mathbb{Z}_p =$ inv. lim. $\mathbb{Z}/p^\nu\mathbb{Z}$ is dual to $\mathbb{Q}_p/\mathbb{Z}_p =$ dir. lim. $p^{-\nu}\mathbb{Z}/\mathbb{Z}$; similarly, $\hat{\mathbb{Z}}$ is dual to $\mathbb{Q}/\mathbb{Z}$.

If $f : \mathbb{Q}_{discrete} \to \mathbb{R}$ is the inclusion map then $f' : \mathbb{R}' = \mathbb{R} \to \mathbb{Q}'$. Here $\mathbb{Q}'$, the dual of the discrete group $\mathbb{Q}$, is the solenoidal compactification of $\mathbb{R}$, i.e.,the inverse limit of the groups $\mathbb{R}/n\mathbb{Z}$ relative to the canonical maps $\mathbb{R}/m\mathbb{Z} \to \mathbb{R}/n\mathbb{Z}$ for $m|n$, because $\mathbb{Q}$ is the direct limit (union) of the groups $n^{-1}\mathbb{Z}/\mathbb{Z}$.

The direct product of compact groups G_i is dual to the direct sum (coproduct) of their discrete duals G_i'. This suggests that, more generally, the restricted direct product (rdp) of groups G_v relative to some open compact subgroups H_v, is dual to the rdp of the duals G_v' rel. to the subgroups $H^\perp$. This is a nice exercise. See ANT p.324. If $< \ , \ >_v$ is the pairing between G_v and G_v' then the pairing of the rdp's is simply $< x, y >= \prod_v < x_v, y_v >_v$ for $x = (x_v), y = (y_v)$. Note that the product is finite in the sense that almost all of the factors are 1, because $< H_v, H_v^\perp >_v = 1$.

3, Haar measure.

Let $L(G)$ denote the vector space of continuous complex valued functions f on G with compact support. A Haar measure on G is a non zero linear map $L(G) \to \mathbb{C}$ which is invariant under translation and which is positive on functions whose values are ≥ 0. Such a map is unique up to a positive constant and is given by an integral $f \to \int_G f(x)d\mu(x)$ where μ is a Radon measure on G for which compact sets have finite measure and non-empty open sets have measure > 0. If G is compact one can fix the constant in μ by requiring $\mu(G) = 1$. If G is discrete one can fix it so that each point gets measure 1; then $\int_G f(x)d\mu(x) = \sum_{x \in G} f(x)$. We call these normalizations 'canonical' if G is compact or discrete but not finite. Alas, if G is finite $\neq 1$, these two normalizations do not agree. One must choose between $\mu(G) = 1, \mu(G) = |G|$, or perhaps compromise with $\mu(G) = \sqrt{|G|}$.

4. Examples of Haar measures.

We will chose the following measures on the additive groups of local fields.
$K_v = \mathbb{R}$ Lebesgue measure, $\mu_v([0, 1]) = 1$.
$K_v = \mathbb{C}$ $2\times$ Lebesgue measure, μ_v(unit square)=2.
K_v non-archimedian. Normalize so that $\mu_v(A_v) = 1$

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If $G = G_1 \times G_2$ and μ_1, μ_2 are Haar measures on G_1, G_2, then the product measure $\mu = \mu_1 \times \mu_2$ is a Haar measure on G. More generally, if H is a closed subgroup of G, and μ_H is a Haar measure on H and $\mu_{G/H}$ one on G/H, then there is a Haar measure μ on G such that

$$\int_G f(x)d\mu(x) = \int_{G/H} (\int_H f(x + h)d\mu_H(h))d\mu_{G/H}(\bar{x}),$$

where the inner integral, which depends only on $\bar{x} := x \bmod H$ is viewed as a function on G/H. We write $\mu = \mu_{G/H}\mu_H$. It is a theorem that for $f \in L^1(G)$ the inner integral is defined for almost all $\bar{x}$ and yields a function in $L^1(G/H)$ and the equality holds, just as in the Fubini theorem.

Examples: The Lebesgue measure on $\mathbb{R}$ is the 'product' in the sense of the preceding paragraph of the canonical measure on the compact quotient $\mathbb{R}/\mathbb{Z}$ and the canonical measure on the discrete subgroup $\mathbb{Z}$. For a non-archimedean local field K_v our choice above of measure on the additive group of K_v is the product of the canonical measure on the discrete quotient K_v/A_v and the canonical measure on the compact subgroup A_v.

In the rdp situation $G = \overline{\prod} G_v$ discussed above, if we are given Haar measures μ_v on the factors G_v such that $\mu_v(G_v) = 1$ for almost all v, then there is a product measure on G such that, given $f_v \in L^1(G_v)$ such that f_v is the characteristic function of h_v for almost all v, we have, defining $f(x) = \prod_v f_v(x_v)$,

$$\int_G f(x) d\mu(x) = \prod_v \int_{G_v} f_v(x_v) d\mu_v(x_v),$$

a finite product in that almost all factors are 1. This is a simple exercise. See ANT p.325. We write $\mu = \prod_v \mu_v$.

For each gobal field K we now have a measure on the adele group $\mathbb{A}_K$, namely, the product over the places v of K of the μ_v defined above on the local fields K_v.

5. Fourier transform

Let $f \in L^1(G)$ and let μ be a Haar measure on G.

Definition: The Fourier transform of f relative to μ is the function $\hat{f}$ on G' defined by

$$\hat{f}(y) = \int_G f(x) \overline{< x, y >} d\mu(x).$$

Exercise 1: Prove that $\hat{f}$ is continuous on G'.

Theorem (Inversion Formula) There exists a (unique) μ' on G' such that if both f is continuous and $f' \in L_1(G')$, then

$$f(x) = \int_{G'} \hat{f}(y) < x, y > d\mu'(y) \quad (= \hat{\hat{f}}(-x)).$$

Definition: We call μ' the dual measure to μ.

Exercise 2: For $c > 0$, $(c\mu)' = c^{-1}\mu'$. Also, $(\mu')' = \mu$.

Exercise 3: If G is compact and $\mu(G) = 1$, then μ' is the measure on G' for which each point has measure 1.

Exercise 4: If $g(x) = f(x + x_0)$ then $\hat{g}(y) =< x_0, y > \hat{f}(y)$.

Exercise 5. Suppose α is an automorphism of G and $g(x) = f(\alpha x)$. Let $|\alpha|$ denote the constant such that $\mu(\alpha U) = |\alpha|\mu(U)$ for measurable sets U. Then

$$\hat{g}(y) = |\alpha|^{-1} \hat{f}((\alpha')^{-1} y).$$

6. Examples of Fourier Transforms.

Exercise 6: Suppose H is an open compact subgroup of G, and f is the characteristic function of H. Then $\hat{f}$ is $\mu(H)$ times the characteristic function of $\hat{H}$.

Exercise 7: If $G = \mathbb{R} = G'$ via $< x, y >= \exp(2\pi ixy)$ and $f(x) = \exp(-\pi x^2)$, then $\hat{f}(y) = \exp(-\pi y^2)$. That is, $\hat{f} = f$.

Exercise 8: Suppose the Fourier series of a function f on $\mathbb{R}$ such that $f(x+1) = f(x)$ is the series $\sum_{n\in\mathbb{Z}} c(n)\exp(2\pi inx)$. Show that the function $c : \mathbb{Z} \to \mathbb{C}$ is the Fourier transform of the function F on $\mathbb{R}/\mathbb{Z}$ such that $F(x + \mathbb{Z}) = f(x)$, relative to the measure on $\mathbb{R}/\mathbb{Z}$ giving that compact group measure 1, if we identify $\mathbb{Z}$ with $(\mathbb{R}/\mathbb{Z})'$ by letting an integer n correspond to the character $x + \mathbb{Z} \mapsto \exp(2\pi inx)$.

Exercise 9: Let G be the rdp of the G_v' relative to some open compact H_v's. Let $f_v \in L^1(G_v)$ for all v and suppose f_v is the characteristic function of H_v for almost all v. Let $\mu = \prod_v \mu_v$ be a Haar measure on G as above. Then the Fourier transform of $f(x) = \prod_v f_v(x_v)$ is $\prod_v \hat{f}_v$ in the obvious sense, and if $f_v, \hat{f}_v$ satisfy the condition of the inversion formula Theorem above for each v, then inversion holds for $f, \hat{f}$.

7. Adeles (Finally some number theory!)

Let K be a global field. Let $\mathbb{A} = \mathbb{A}_K$ be its adele ring, the rdp of the K_v relative to the subrings A_v for finite v. Recall that K is canonically imbedded as a discrete subfield of $\mathbb{A}$

Theorem: Let λ be a non-trivial character of $\mathbb{A}$ which is trivial on K. In other words, $0 \neq \lambda \in K^\perp \subset \mathbb{A}'$. Then the pairing $< x, y >= \lambda(xy)$ identifies $\mathbb{A}'$ with $\mathbb{A}$, and $K^\perp$ with K. Looked at another way, the statement is that $\mathbb{A}'$ is a free $\mathbb{A}$-module of rank 1 with basis $\{\lambda\}$, and the map $y \mapsto y\lambda$ from $\mathbb{A}$ to $\mathbb{A}'$ is a homeomorphism as well as an isomorphism of $\mathbb{A}$-modules, carrying K onto $K^\perp$.

Proof: If the theorem is true for one such λ it is true for all, because the last statement of the theorem shows that $K^\perp$ is a 1-dimensional vector space over K, and it is obvious that if $\alpha \in K^*$ and the theorem is true for λ, then it is true for $\alpha\lambda$.

Next, if the theorem is true for K it is true for an extension field $L \supset K$. This follows easily from

Lemma: For global fields $K \subset L$ we denote places of K by v and places of L by w. Then $K_v \otimes_K L = \prod_{w|v} L_w$ for all v, and for finite v, $A_v \otimes_{A_K} A_L = \prod_{w|v} A_w$. Hence, $\mathbb{A}_K \otimes_K L = \mathbb{A}_L$.

Proof. postponed (see Lecture3, section 1)

Number field case: Here there is a canonical choice of λ. For $\mathbb{Q}$ we define as in lecture 1, characters λ_v as follows:

$$\lambda_\infty(x) = \exp(-2\pi ix), \qquad \lambda_p(\mathbb{Z}_p) = 0, \quad \lambda_p(1/p^\nu) = \exp(2\pi i/p^\nu)$$

(It is easy to see that there do exist unique such λ_p's.) Putting $\lambda = \prod_v \lambda_v$ we get a character on $\mathbb{A}_\mathbb{Q}$ and check that it is trivial on $\mathbb{Q}$ because $\lambda(1/p^\nu) = \exp(-2\pi i/p^\nu)\exp(2\pi i/p^\nu) = 1$. Clearly, λ_p identifies ${\mathbb{Q}_p}'$ with $\mathbb{Q}_p$ and ${\mathbb{Z}_p}^\perp$ with $\mathbb{Z}_p$ for all p. Also, λ_∞ identifies $\mathbb{R}'$ with $\mathbb{R}$. Hence λ identifies ${\mathbb{A}_\mathbb{Q}}'$ with $\mathbb{A}_Q$.

We must now show that $\mathbb{Q}^\perp = \mathbb{Q}\lambda$. For this we put $\mathbb{A}(\infty) = \mathbb{R} \times \prod_p \mathbb{Z}_p$ and note that $\mathbb{A}_Q = \mathbb{Q} + \mathbb{A}(\infty)$, and $\mathbb{Q} \cap \mathbb{A}(\infty) = \mathbb{Z}$. Hence $\mathbb{A}_\mathbb{Q}/\mathbb{Q} = \mathbb{A}(\infty)/\mathbb{Z}$. This quotient group is the image of the compact set $[0,1] \times \prod_p \mathbb{Z}_p$, so is compact. Hence $\mathbb{Q}^\perp$ is discrete. It is a vector space over $\mathbb{Q}$ containing $\mathbb{Q}$. and $\mathbb{Q}^\perp/\mathbb{Q}$ is discrete and compact therefore finite. Hence $\mathbb{Q}^\perp = \mathbb{Q}$.

That proves the theorem for $\mathbb{Q}$, and by the Lemma, the theorem and the compactness of $\mathbb{A}_K/K$ follows for all number fields K.

In the function field case we will identify the space $K^\perp$ with the space $\Omega^1_{K/\mathbb{F}_p}$ of rational differentials of K...

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