

**IMAGE AND KERNEL**

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Homework: Section 3.1: 12,22,34,44,54,38\*,48\*,46\*

IMAGE. If  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a linear transformation, then  $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^m\}$  is called the **image** of  $T$ . If  $T(\vec{x}) = A\vec{x}$ , then the image of  $T$  is also called the image of  $A$ . We write  $\text{im}(A)$  or  $\text{im}(T)$ .

EXAMPLES.

- 1) If  $T(x, y, z) = (x, y, 0)$ , then  $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ . The image of  $T$  is the  $x - y$  plane.
- 2) If  $T(x, y) = (\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$  is a rotation in the plane, then the image of  $T$  is the whole plane.
- 3) If  $T(x, y, z) = x + y + z$ , then the image of  $T$  is  $\mathbf{R}$ .

SPAN. The **span** of vectors  $\vec{v}_1, \dots, \vec{v}_k$  in  $\mathbf{R}^n$  is the set of all combinations  $c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ , where  $c_i$  are real numbers.

PROPERTIES.

The image of a linear transformation  $\vec{x} \mapsto A\vec{x}$  is the span of the column vectors of  $A$ .  
The image of a linear transformation contains 0 and is closed under addition and scalar multiplication.

KERNEL. If  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a linear transformation, then the set  $\{x \mid T(x) = 0\}$  is called the **kernel** of  $T$ . If  $T(\vec{x}) = A\vec{x}$ , then the kernel of  $T$  is also called the kernel of  $A$ . We write  $\text{ker}(A)$  or  $\text{ker}(T)$ .

EXAMPLES. (The same examples as above)

- 1) The kernel is the  $z$ -axes. Every vector  $(0, 0, z)$  is mapped to 0.
- 2) The kernel consists only of the point  $(0, 0, 0)$ .
- 3) The kernel consists of all vector  $(x, y, z)$  for which  $x + y + z = 0$ . The kernel is a plane.

PROPERTIES.

The kernel of a linear transformation contains 0 and is closed under addition and scalar multiplication.

IMAGE AND KERNEL OF INVERTIBLE MAPS. A linear map  $\vec{x} \mapsto A\vec{x}$ ,  $\mathbf{R}^n \mapsto \mathbf{R}^n$  is invertible if and only if  $\text{ker}(A) = \{0\}$  if and only if  $\text{im}(A) = \mathbf{R}^n$ .

HOW DO WE COMPUTE THE IMAGE? The column vectors of  $A$  span the image. We will see later that the columns with leading ones alone span already the image.

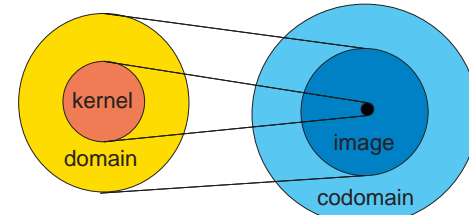
EXAMPLES. (The same examples as above)

- 1)  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  span the image.
- 2)  $\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \end{bmatrix}$  and  $\begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix}$  span the image.
- 3) The 1D vector  $[1]$  spans the image.

HOW DO WE COMPUTE THE KERNEL? Just solve  $A\vec{x} = \vec{0}$ . Form  $\text{rref}(A)$ . For every column without leading 1 we can introduce a free variable  $s_i$ . If  $\vec{x}$  is the solution to  $A\vec{x}_i = 0$ , where all  $s_j$  are zero except  $s_i = 1$ , then  $\vec{x} = \sum_j s_j \vec{x}_j$  is a general vector in the kernel.

EXAMPLE. Find the kernel of the linear map  $\mathbf{R}^3 \rightarrow \mathbf{R}^4$ ,  $\vec{x} \mapsto A\vec{x}$  with  $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$ . Gauss-Jordan

elimination gives:  $B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . We see one column without leading 1 (the second one). The equation  $B\vec{x} = 0$  is equivalent to the system  $x + 3y = 0, z = 0$ . After fixing  $z = 0$ , can chose  $y = t$  freely and obtain from the first equation  $x = -3t$ . Therefore, the kernel consists of vectors  $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ . In the book, you have a detailed calculation, in a case, where the kernel is 2 dimensional.



WHY DO WE LOOK AT THE KERNEL?

- It is useful to understand linear maps. To which degree are they non-invertible?
- Helpful to understand quantitatively how many solutions a linear equation  $Ax = b$  has. If  $x$  is a solution and  $y$  is in the kernel of  $A$ , then also  $A(x + y) = b$ , so that  $x + y$  solves the system also.

WHY DO WE LOOK AT THE IMAGE?

- A solution  $Ax = b$  can be solved if and only if  $b$  is in the image of  $A$ .
- Knowing about the kernel and the image is useful in the similar way that it is useful to know about the domain and range of a general map and to understand the graph of the map.

In general, the abstraction helps to understand topics like error correcting codes (Problem 53/54 in Bretschers book), where two matrices  $H, M$  with the property that  $\text{ker}(H) = \text{im}(M)$  appear. The encoding  $x \mapsto Mx$  is robust in the sense that adding an error  $e$  to the result  $Mx \mapsto Mx + e$  can be corrected:  $H(Mx + e) = He$  allows to find  $e$  and so  $Mx$ . This allows to recover  $x = PMx$  with a projection  $P$ .

PROBLEM. Find  $\text{ker}(A)$  and  $\text{im}(A)$  for the  $1 \times 3$  matrix  $A = [5, 1, 4]$ , a row vector.

ANSWER.  $A \cdot \vec{x} = A\vec{x} = 5x + y + 4z = 0$  shows that the kernel is a plane with normal vector  $[5, 1, 4]$  through the origin. The image is the codomain, which is  $\mathbf{R}$ .

PROBLEM. Find  $\text{ker}(A)$  and  $\text{im}(A)$  of the linear map  $x \mapsto v \times x$ , (the cross product with  $v$ ).

ANSWER. The kernel consists of the line spanned by  $v$ , the image is the plane orthogonal to  $v$ .

PROBLEM. Fix a vector  $w$  in space. Find  $\text{ker}(A)$  and image  $\text{im}(A)$  of the linear map from  $\mathbf{R}^6$  to  $\mathbf{R}^3$  given by  $x, y \mapsto [x, v, y] = (x \times y) \cdot w$ .

ANSWER. The kernel consist of all  $(x, y)$  such that their cross product orthogonal to  $w$ . This means that the plane spanned by  $x, y$  contains  $w$ .

PROBLEM Find  $\text{ker}(T)$  and  $\text{im}(T)$  if  $T$  is a composition of a rotation  $R$  by 90 degrees around the  $z$ -axes with with a projection onto the  $x$ - $z$  plane.

ANSWER. The kernel of the projection is the  $y$  axes. The  $x$  axes is rotated into the  $y$  axes and therefore the kernel of  $T$ . The image is the  $x$ - $z$  plane.

PROBLEM. Can the kernel of a square matrix  $A$  be trivial if  $A^2 = \mathbf{0}$ , where  $\mathbf{0}$  is the matrix containing only 0?

ANSWER. No: if the kernel were trivial, then  $A$  were invertible and  $A^2$  were invertible and be different from  $\mathbf{0}$ .

PROBLEM. Is it possible that a  $3 \times 3$  matrix  $A$  satisfies  $\text{ker}(A) = \mathbf{R}^3$  without  $A = \mathbf{0}$ ?

ANSWER. No, if  $A \neq \mathbf{0}$ , then  $A$  contains a nonzero entry and therefore, a column vector which is nonzero.

PROBLEM. What is the kernel and image of a projection onto the plane  $\Sigma : x - y + 2z = 0$ ?

ANSWER. The kernel consists of all vectors orthogonal to  $\Sigma$ , the image is the plane  $\Sigma$ .

PROBLEM. Given two square matrices  $A, B$  and assume  $AB = BA$ . You know  $\text{ker}(A)$  and  $\text{ker}(B)$ . What can you say about  $\text{ker}(AB)$ ?

ANSWER.  $\text{ker}(A)$  is contained in  $\text{ker}(BA)$ . Similar  $\text{ker}(B)$  is contained in  $\text{ker}(AB)$ . Because  $AB = BA$ , the kernel of  $AB$  contains both  $\text{ker}(A)$  and  $\text{ker}(B)$ . (It can be bigger:  $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .)

PROBLEM. What is the kernel of the partitioned matrix  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  if  $\text{ker}(A)$  and  $\text{ker}(B)$  are known?

ANSWER. The kernel consists of all vectors  $(\vec{x}, \vec{y})$ , where  $\vec{x} \in \text{ker}(A)$  and  $\vec{y} \in \text{ker}(B)$ .