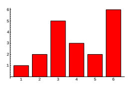
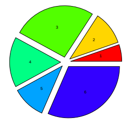
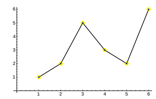
FUNCTION SPACES	Math 21b, O. Knill

Homework: Section 4.1: 6-8,9-11,36,48,58,44*,12*-15*

FROM VECTORS TO FUNCTIONS AND MATRICES. Vectors can be displayed in different ways:

$$\begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 2 \\ 6 \end{bmatrix}$$

The values $(i, \vec{v}_i)$ can be interpreted as the graph of a **function** $f : 1,2,3,4,5,6 \to \mathbf{R}$, where $f(i) = \vec{v}_i$.

Also matrices can be treated as functions, but as a function of two variables. If M is a 8×8 matrix for example, we get a function $f(i,j) = [M]_{ij}$ which assigns to each square of the 8×8 checkerboard a number.

LINEAR SPACES. A space X which contains 0, in which we can add, perform scalar multiplications and where basic laws like commutativity, distributivity and associativity hold, is called a **linear space**.

BASIC EXAMPLE. If A is a set, the space X of all functions from A to R is a linear space. Here are three important special cases:

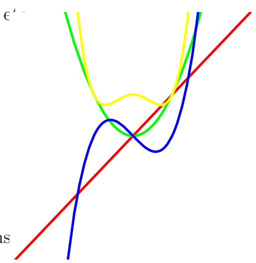
EUCLIDEAN SPACE: If $A = \{1,2,3,...,n\}$, then X is R^n itself.
FUNCTION SPACE: If A is the real line, then X is a the space of all functions in one variable.
SPACE OF MATRICES: If A is the set

$$\begin{array}{cccc} (1,1) & (1,2) & \dots & (1,m) \\ (2,1) & (2,2) & \dots & (2,m) \\ \dots & \dots & \dots & \dots \\ (n,1) & (n,2) & \dots & (n,m) . \end{array}$$

Then X is the space of all $n \times m$ matrices.

EXAMPLES.

- The n-dimensional space R^n.
- linear subspaces of R^n like the trivial space $\{0\}$, lines or planes e
- M_n, the space of all square $n \times n$ matrices.
- P_n, the space of all polynomials of degree n.
- The space P of all polynomials.
- C^∞, the space of all smooth functions on the line
- C^0, the space of all continuous functions on the line.
- $C^\infty(R^3, R^3)$ the space of all smooth vector fields in three dimens
- C^1, the space of all differentiable functions on the line.
- $C^\infty(\mathbf{R}^3)$ the space of all smooth functions in space.
- L^2 the space of all functions for which $\int_\infty^\infty f^2(x)\, dx < \infty$.



ZERO VECTOR. The function f which is everywhere equal to 0 is called the **zero function**. It plays the role of the zero vector in R^n. If we add this function to an other function g we get $0 + g = g$.

Careful, the **roots** of a function have nothing to do with the zero function. You should think of the roots of a function like as zero entries of a vector. For the zero vector, all entries have to be zero. For the zero function, all values $f(x)$ are zero.

CHECK: For subsets X of a function space, of for a ssubset of matrices R^n, we can check three properties to see whether the space is a linear space:

i) if x, y are in X, then $x + y$ is in X.
ii) If x is in X and λ is a real number, then λx is in X.
iii) 0 is in X.

WHICH OF THE FOLLOWING ARE LINEAR SPACES?

- The space X of all polynomials of the form $f(x) = ax^3 + bx^4 + cx^5$
- The space X of all continuous functions on the unit interval $[-1,1]$ which are zero at -1 and 1. It contains for example the function $f(x) = x^2 - |x|$.
- The space X of all smooth periodic functions $f(x+1) = f(x)$. Example $f(x) = \sin(2\pi x) + \cos(6\pi x)$.
- The space $X = \sin(x) + C^\infty(\mathbf{R})$ of all smooth functions $f(x) = \sin(x) + g$, where g is a smooth function.
- The space X of all trigonometric polynomials $f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + ... + a_n \sin(nx)$.
- The space X of all smooth functions on $\mathbf{R}$ which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.
- The space X of all continuous functions on $\mathbf{R}$ which satisfy $f(2) = 0$ and $f(10) = 0$.
- The space X of all smooth functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 0$.
- The space X of all continuous functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 1$.
- The space X of all smooth functions on $\mathbf{R}$ of compact support: for every f, there exists an interval I such that $f(x) = 0$ outside that interval.
- The space X of all smooth functions on $\mathbf{R}^2$.

If you have taken multivariable calculus you might like the following examples:

- The space X of all vector fields (P,Q) in the plane, for which the curl $Q_x - P_y$ is zero everywhere.
- The space X of all vector fields (P,Q,R) in space, for which the divergence $P_x + Q_y + R_z$ is zero everywhere.
- The space X of all vector fields (P,Q) in the plane for which the line integral $\int_C F \cdot dr$ along the unit circle is zero.
- The space X of all vector fields (P,Q,R) in space for which the flux through the unit sphere is zero.
- The space X of all functions $f(x,y)$ of two variables for which $\int_0^1 \int_0^1 f(x,y)\, dxdy = 0$.