ODE COOKBOOK Math 21b, O. Knill

$$x' - \lambda x = 0 \qquad x(t) = Ce^{\lambda t}$$

This first order ODE is by far the most important differential equation. A linear system of differential equation $x'(t) = Ax(t)$ reduces to this after diagonalization. We can rewrite the differential equation as $(D - \lambda)x = 0$. That is x is in the kernel of $D - \lambda$. An other interpretation is that x is an eigenfunction of D belonging to the eigenvalue λ. This differential equation describes exponential growth or exponential decay.

$$x'' + k^2 x = 0 \qquad x(t) = C_1 \cos(kt) + C_2 \sin(kt)$$

This second order ODE is by far the second most important differential equation. Any linear system of differential equations $x''(t) = Ax(t)$ reduces to this after diagonalization. We can rewrite the differential equation as $(D^2 + k^2)x = 0$. That is x is in the kernel fo $D^2 + k^2$. An other interpretation is that x is an eigenfunction of D^2 belonging to the eigenvalue $-k^2$. This differential equation describes oscillations or waves.

OPERATOR METHOD. A general method to find solutions to $p(D)x = g$ is to factor the polynomial $p(D) = (D - \lambda_1) \cdots (D - \lambda_n)x = g$, then invert each factor to get

$$x = (D - \lambda_n)^{-1} ..... (D - \lambda_1)^{-1} g$$

where

$$(D - \lambda)^{-1} g = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s) \, ds$$

COOKBOOK METHOD. The operator method always works but it can produce a considerable amount of work. Engineers therefore rely also on cookbook recipes. The solution of an inhomogeneous differential equation $p(D)x = g$ is found by first finding the **homogeneous solution** x_h which is the solution to $p(D)x = 0$. Then a particular solution h_p of the system $p(D)x = g$ found by an educated guess. This method is often much faster but it requires to know the "recipes". Fortunately, it is quite easy: as a rule of thumb: feed in the same class of functions which you see on the right hand side and if the right hand side should contain a function in the kernel of $p(D)$, try with a function multiplied by t. The general solution of the system $p(D)x = g$ is $x = x_h + x_p$.

FINDING THE HOMOGENEOUS SOLUTION. $p(D) = (D - \lambda_1)(D - \lambda_2) = D^2 + bD + c$. The next table covers all cases for homogeneous second order differential equations $x'' + bx' + c = 0$.

$\lambda_1 \neq \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
$ik = \lambda_1 = -\lambda_2$ imaginary	$C_1 \cos(kt) + C_2 \sin(kt)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$C_1 e^{at} \cos(kt) + C_2 e^{at} \sin(kt)$

FINDING AN INHOMOGENEOUS SOLUTION. This can be found by applying the operator inversions with $C = 0$ or by an educated guess. to solve $x'' + bx' + c = g(t)$. If $b = c = 0$, then $x'' = g(t)$ can be solved by integrating twice, otherwise, check with the following table:

$g(t) = a$ constant	$x(t) = A$ constant
$g(t) = at + b$	$x(t) = At + B$
$g(t) = at^2 + bt + c$	$x(t) = At^2 + Bt + C$
$g(t) = a\cos(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\sin(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\cos(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = a\sin(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = ae^{bt}$	$x(t) = Ae^{at}$
$g(t) = ae^{bt}$ with $p(D)g = 0$	$x(t) = Ate^{at}$
$g(t) = q(t)$ polynomial	$x(t) =$ polynomial of same degree