# A bit of review for Math 21b Midterm 1 on Wednesday, March 6, 2002

Things to keep in mind

Row – the set of elements comprising a horizontal line in a matrix
Column – the set of elements comprising a vertical line in a matrix
Entry – a number or variable occupying a single position in a matrix
Row vector – the ordered set of elements in a specific row
Vector – the ordered set of elements in a specific column
Number/scalar – interchangeable terms

Coefficient matrix – the matrix whose entries are the coefficients of a system of linear equations, where each row represents a different equation and each column represents the coefficients of a different variable

Augmented matrix – adding a column vector to the right side of a coefficient matrix where that column vector represents the vector b in the equation Ax = b

Gaussian elimination – a method of reducing any matrix to a form readily displaying much information about the transformation represented in the original matrix

rref – "reduced row echelon form"; a matrix where each row's first nonzero entry is a 1 (called a "leading 1"), and where each column containing a leading 1 contains 0's everywhere else; the form of a matrix that Gaussian elimination leads to

Consistent/inconsistent – the situation of either having a solution to a system of linear equations, or not having any (respectively); there can be 0, 1, or infinitely many solu tions to a system of equations

Leading "1" (as opposed to a leading variable) – when dealing with a coefficient matrix, all the leading 1's are leading variables, but when dealing with an augmented matrix a leading 1 in the last column does not represent any variable, but indicates an incon sistent system; if all the variables are leading 1's then there is a unique solution to the equation, but if any are nonleading variables then there are infinitely many solutions

Rank – the number of leading 1's in an rref'd matrix, not necessarily the number of leading variables (see above for the distinction)

Linear combination – something is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$ if it can be written as $k_1v_1 + k_2v_2 + \ldots + k_nv_n$ where $k_1, k_2, \ldots, k_n$ are constants

Vector/matrix – a matrix is a rectangular array of numbers; a matrix is often described in terms of its dimension as mxn where m is the number of rows (number of entries in each column) and n is the number of columns (number of entries in each row); a vector is simply an mx1 matrix

$\mathbf{R}^n$ – is the space defined by ordered "n-tuples" of real numbers (e.g. $(x_1, x_2, \ldots, x_n)$ )

Linear transformation – a function T from $\mathbf{R}^n$ to $\mathbf{R}^m$ where there is an mxn matrix A such that T(x) = A(x); in other words, a function whose operation on x can be written as some matrix A times the n-tuple x

Inverse linear transformation (when it exists) – a transformation that "undoes" what ever the transformation T does to a vector, a way of getting back to the original vector; a matrix A (and hence a transformation T) has an inverse if and only if the transformation matrix A is an nxn matrix (it's square) and rank(A) = n

Identity matrix $I_n$ – the nxn matrix consisting of all 0's except along the main diagonal which is all 1's

Zero matrix – any matrix consisting entirely of 0's

Geometric representation – a way of interpreting geometrically what a matrix does to a vector; in other words, think back to the idea of transformation and ask "what happens to a vector x after I transform it?"

Standard (unit) vectors $e_1$, $e_2$, … – these are vectors that represent the direction of each coordinate axis; in other words, in 3-dimensional space, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$; note that $a_1e_1 + a_2e_2 + \ldots + a_ne_n$ is the vector $(a_1, a_2, \ldots, a_n)$ and the column vectors of a matrix A are equal to $A(e_1)$, $A(e_2)$, etc.

Linear characterization of linear transformation – determining exactly what sort of geometric interpretation to associate with a given transformation; the possibilities are:

Dilation – changing the magnitude of all vectors by a constant factor but maintaining their directions

Rotation – changing the direction of vectors by altering the angle with an axis by a constant amount, but maintaining each vector's magnitude

Rotation and Dilation – changing the magnitude and direction of vectors as described above

Shear – a transformation that moves each point in the direction of some line through the origin

Orthogonal projection – drawing a perpendicular line from the vector to some specific line and making the image of the original vector wherever that perpendicular intersects the line

How to decide when a linear transformation is invertible – when each output of the transformation came from a unique input (rank = n)

How to find an inverse – augment the matrix A with the identity matrix of the same size and row reduce A, making sure to apply the same operations to the identity matrix as to A; when A is reduced to the identity (which must be possible in order for A to have an inverse, this is equivalent to saying rank(A) = n), the matrix on the right hand side, where the identity was originally, is the inverse of A, denoted $A^{-1}$

Composition of functions – the process of first applying one function (or transformation) and then applying a second function (or transformation) to the outcome of the first; when writing compositions work backwards from right to left $f \circ g(x)$ means first do g, then f; this can also be written as $f(g(x))$

How to compute a matrix product – first be aware that not all matrices can be multiplied, they need to be of compatible sizes, namely an mxn matrix can only be multiplied on the right by an nxp matrix, and the outcome will be an mxp matrix; to determine the entry in the ith row and jth column of the product AB, consider the ith row of A and the jth column of B and add the products of their corresponding terms (multiply the first terms in each, add this to the product of the second terms in each, add this to the product of the third terms in each, etc.)

Properties of matrix multiplication – associativity A(BC) = (AB)C, but not always commutativity (AB = BA)

$(AB)^{-1} = B^{-1}A^{-1}$

Span – the span of a set of vectors is the set of all linear combinations of those vectors

Image of a linear transformation – this is equivalent to the span of the column vectors of the matrix A for the transformation T, and occasionally called the column space of A; it is all possible outputs of the transformation; it is a subset of the *range*; Im(A) has the following properties:

1. The zero vector is always in the image because A(0) = 0
2. If $v_1$ and $v_2$ are in Im(A) then ($v_1 + v_2$) is in Im(A)
3. If v is in Im(A) then kv is in Im(A), where k is any scalar

Kernel of a linear transformation – also called the nullspace; this is the set of all vectors which get mapped by T (or equivalently A) to the zero vector; it is a subset of the *domain*; ker(A) has the following properties:

1. The zero vector is always in the kernel because A(0) = 0
2. If $v_1$ and $v_2$ are in Ker(A) then ($v_1 + v_2$) is in Ker(A)
3. If v is in Ker(A) then kv is in Ker(A), where k is any scalar

The invertible case – if f, a function from X to Y, is invertible, then Im(f) = Y; an nxn matrix A is invertible if and only if Ax = b has a unique solution for all vectors b, rref(A) = I, rank(A) = n, Im(A) = $\mathbf{R}^n$ and Ker(A) = {0} (does not contain anything other than the zero vector), the column vectors of A are linearly independent

Space – a mathematical term that indicates structure and dimension

Subspace – a subset W of a space such that 0 is in W, if $w_1$ and $w_2$ are in W then $w_1 + w_2$ is in W, and if w is in W then kw is in W for any scalar k; note: {0} and $\mathbf{R}^n$ are always subspaces of $\mathbf{R}^n$

Linear dependence – a set of vectors $v_1, \ldots, v_n$ is said to be linearly dependent if one of the vectors can be written as a linear combination of the rest of the vectors; equivalently, a set of vectors is linearly dependent if some nontrivial linear combination of them (nontrivial meaning that not all the coefficients are 0) equals the zero vector; note that if v is a linear combination of $v_1, \ldots, v_n$ then Span(v, $v_1, \ldots, v_n$) = Span ($v_1, \ldots, v_n$)

Linear independence – a set of vectors $v_1, \ldots, v_n$ is said to be linearly independent if none of the vectors can be written as a linear combination of any of the others

There cannot be more linearly independent vectors than there are spanning vectors.

Equivalently, if the space is spanned by q vectors, than any set of more than q vectors is lin early dependent.

Basis – a set of vectors that are linearly independent and span the space; all bases of a particular space consist of the same number of vectors

Dimension – the dimension of a subspace V of $\mathbf{R}^n$ is the number of vectors in a basis of V; if dim(V) = d, then:

It is impossible to find more than d linearly independent vectors in V

No less than d vectors will span V

If d vectors in V are linearly independent than they form a basis of V

If d vectors span V then they form a basis of V

dim(ker(A)) = n – rank(A)

rank(A) = dim(im(A))

dim(ker(A)) + dim(im(A)) = n

Review procedures for finding bases for the kernel and for the image of a the image of a transformation

Pivot column – if the ith column of rref(A) contains a leading 1, then the ith column of A is called a pivot column

The columns of an nxn matrix A for a basis for $\mathbf{R}^n$ if and only if A is invertible

READ and KNOW Summary 3.3.11 on page 170

Coordinates in $\mathbf{R}^n$ : The coordinates of a vector x in a basis $B = v_1 , v_2 ,\ldots, v_m$ are related to the coordinates of x in the standard basis by $x = S[x]_B$ , where $S = [v_1\ v_2\ \ldots\ v_m]$ is an n by m matrix.

We can write the matrix of a linear transformation T with respect to a basis $B = v_1 , v_2 ,\ldots,v_n$ as

$$B = [\ [T(v_1)]_B\ [T(v_2)]_B \ldots [T(v_n)]_B\ ] .$$

Furthermore, the matrix B of T with respect to *B* is related to the matrix A of T with respect to the standard basis by

$$B = S^{-1}AS$$

where S is as above. Note that B and A are also *similar* as a consequence of this.

Practice Problems

(You certainly do not have to do all of these in order to be prepared for the midterm, but it would not be a bad idea to look through them. Also, all the problems are odd-numbered problems, so their answers should be in the back of the book)

Section 1.1: 15, 17, 25, 31, 37
Section 1.2: 9, 29, 31, 39
Section 1.3: 7, 17, 19, 23, 27, 29, 43, 51
Section 2.1: 7, 11, 13, 33, 45
Section 2.2: 13, 17, 23, 27, 33, 43, 51
Section 2.3: 5, 19, 29, 35, 47
Section 2.4: 7, 17, 23, 35, 47, 67, 71, 81
Section 3.1: 7, 23, 37, 43, 51, 53
Section 3.2: 1, 5, 17, 23, 27, 37, 49
Section 3.3: 7, 19, 21, 27, 29, 33, 41, 45, 53, 61
Section 3.4: 3, 9, 13, 17, 23, 29, 35, 45

More Questions/Problems

1. T is a linear transformation from V to W. Complete the following equations (where u and v are vectors in V and k is any scalar):

a. T (u + v) =

b. T (ku) =

Now state the three properties of the kernel and the image of a transformation, and relate these to the notion of subspace.

2. Consider T : $\mathbf{R}^3$ _ $\mathbf{R}^3$ a matrix transformation given by:

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Show that the Kernel of T is a line through the origin, and find parametric equations of that line.

4. The points (-2, 7), (12, 5), and (4, -11) are points of a circle whose equation has the form $x^2 + y^2 + bx + cy + d = 0$. Write the linear system that can be used to find b, c, and d. After finding the system, show the augmented matrix you would use to solve the system, the reduced row echelon form obtained from this augmented matrix, and the actual solutions.

5. Find the matrix of rotation by an angle theta with respect to the basis (1, 2), (2, 1) and with respect to the basis (cos(t), sin(t)), (-sin(t), cos(t)) (t not equal to theta). Interpret your answer geometrically. Suppose that the matrices you just found were not in terms of these bases, but rather in terms of the standard basis. Would the linear transformations corresponding to these matrices be the same as rotation by theta? If not, describe what these new linear transformations would do.

GOOD LUCK!