BRIEF SOLUTIONS

① F F T T F T T F F F

② (a) $0 < k < 1$

(b) $k > 1$

(c) not possible

(d) $k < 0$

③ f_3, f_1, f_2, f_4

④ (a) ct^n is an eigenvector with eigenvalue n for $n = 0, 1, 2, \ldots$

(b) $T(a_0, a_1, a_2, a_3, \ldots) = (0, a_1, 2a_2, 3a_3, 4a_4, \ldots)$

⑤ (a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(b) $A\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -8 & -4 \\ 1 & -3 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

⑥ (a) $A = \begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$ $\quad f_A(\lambda) = \lambda^2 - 625$

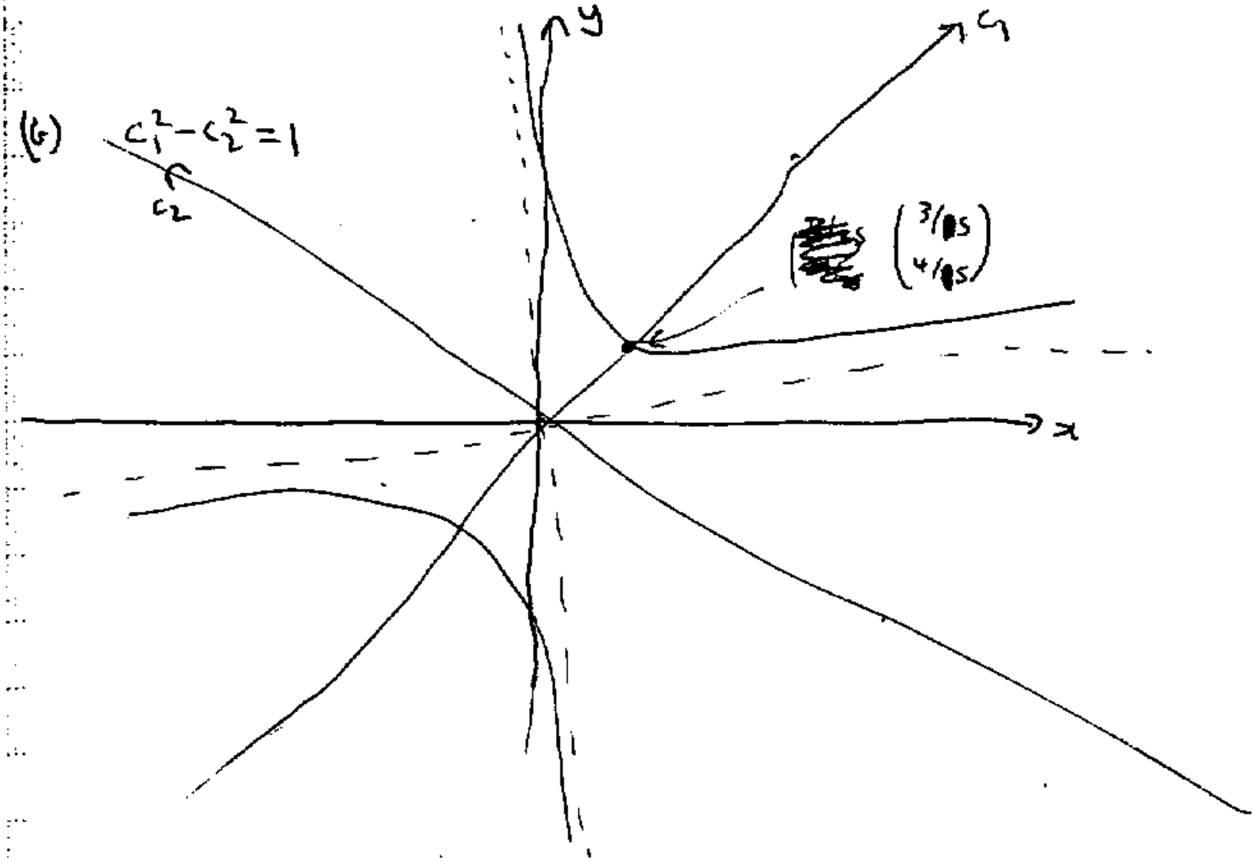
eigenvalues ± 25

$$E_{25} = \ker\begin{pmatrix} -32 & 24 \\ 24 & -18 \end{pmatrix} = \ker\begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$E_{-25} = \ker\begin{pmatrix} 18 & 24 \\ 24 & 32 \end{pmatrix} = \ker\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$\therefore \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$ is an orthonormal eigenbasis.

(b) $c_1^2 - c_2^2 = 1$



⑦ (a) $-x + (2-x)y = 0$ i.e. $y = \frac{x}{2-x}$

$-y + (2-y)x = 0$ i.e. $\frac{x}{x-2} + \left(2 + \frac{x}{x-2}\right)x = 0$

i.e. $x + (3x-4)x = 0$ i.e. $3x(x-1) = 0$

$\therefore (a,b) = (1,1)$

$$\frac{dx}{dt} = -x + (2-x)y = -2(x-1) + (y-1) - (x-1)(y-1)$$

$$\frac{dy}{dt} = -y + (2-y)x = (x-1) - 2(y-1) - (x-1)(y-1)$$

$\therefore \; J = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

(b) $f_J(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$

$\therefore$ eigenvalues $-3, -1$

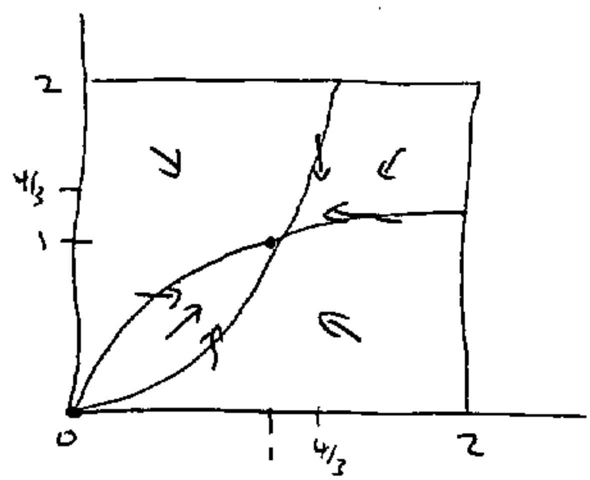
$\therefore$ $(1,1)$ is a stable equilibrium

$E_{-1} = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_{-3} = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenbasis of $\mathbb{R}^2$ for J.

(c)



(d) $(1,1)$ is stable

$(0,0)$ is unstable.

(9) (a) $T(A+B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(A+B) - (A+B)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}A + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B - A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= T(A) + T(B)$

$T(\lambda A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\lambda A) - (\lambda A)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \lambda T(A)$

(b) $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} c & d \\ a & b \end{pmatrix} - \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

$= \begin{pmatrix} c-b & d-a \\ a-d & b-c \end{pmatrix}$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\therefore\ M = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$

(c) $\ker T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c = b,\ a = d \right\} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

$\operatorname{Im} T = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

(d) $T\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = -2\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$T\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = 2\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

$\therefore\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is an eigenbasis of $\mathbb{R}^{2\times 2}$ for T.