Math 21b Midterm II—Solutions
Thursday, April 10, 2003

1. (12 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If A is an $n \times n$ invertible matrix, then $\det(A^T A) > 0$.

Solution. True. $\det(A^T A) = \det(A^T)\det(A) = [\det(A)]^2 > 0$.

T **F** (b) The subset

$$W = \{f \in C^\infty(\mathbb{R}) : f''(x) + f(x) = x^2\}$$

is a subspace of the linear space of all infinitely differentiable functions $C^\infty(\mathbb{R})$.

Solution. False. $f \equiv 0$ is not in W.

T **F** (c) If $\mathbf{v} \in \mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$, then

$$\|\mathbf{v}\|^2 = \|\operatorname{proj}_W \mathbf{v}\|^2 + \|\mathbf{v} - \operatorname{proj}_W \mathbf{v}\|^2.$$

Solution. True. $\operatorname{proj}_W \mathbf{v} \perp \mathbf{v} - \operatorname{proj}_W \mathbf{v}$.

T **F** (d) If $\det(A) \neq \det(B)$, then two $n \times n$ matrices A and B cannot be similar.

Solution. True. Consider the contrapositive. If A and B are similar matrices, then $\det(B) = \det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(A)$.

T **F** (e) The vectors $x+1$, $x-1$, and x^2-1 are linearly dependent in P_2, the polynomials of degree less than or equal to 2.

Solution.
False. Let

$$0 = c_1(x+1) + c_2(x-1) + c_3(x^2-1) = (c_1 - c_2 - c_3) + (c_1 + c_2)x + c_3x^2.$$

We must find a nontrivial solution for

$$\begin{array}{cccl} c_1 & -c_2 & -c_3 & = 0 \\ c_1 & +c_2 & & = 0 \\ & & c_3 & = 0. \end{array}$$

However, it is easy to see that $c_1 = c_2 = c_3 = 0$, and the vectors are linearly independent.

T **F** (f) The determinant of

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 1000 & 7 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1000 \\ 1 & 2 & 3 & 1000 & 4 \end{bmatrix}$$

is positive.

Solution. False. The term 1000^5 is larger than the sum of all of the other terms, and this term is negative.

2. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) (8 points) Find an orthonormal basis for the image of A.

Solution. First,

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

To compute $\mathbf{u}_2$, we first compute

$$\mathbf{a}_2 - \operatorname{proj}_{V_1} \mathbf{a}_2 = \mathbf{a}_2 - (\mathbf{u}_1 \cdot \mathbf{a}_2)\mathbf{u}_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix},$$

and $\|\mathbf{a}_2 - \operatorname{proj}_{V_1} \mathbf{a}_2\| = \sqrt{12}/4$. Thus,

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{a}_2 - \operatorname{proj}_{V_1} \mathbf{a}_2\|}(\mathbf{a}_2 - \operatorname{proj}_{V_1} \mathbf{a}_2) = \begin{bmatrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}.$$

To compute $\mathbf{u}_3$, we first compute

$$\mathbf{a}_3 - \operatorname{proj}_{V_2} \mathbf{a}_3 = \mathbf{a}_3 - [(\mathbf{u}_1 \cdot \mathbf{a}_3)\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{a}_3)\mathbf{u}_2] = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix},$$

and $\|\mathbf{a}_3 - \text{proj}_{V_2}\,\mathbf{a}_3\| = \sqrt{6}/3$. Thus,

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{a}_3 - \text{proj}_{V_2}\,\mathbf{a}_3\|}(\mathbf{a}_3 - \text{proj}_{V_2}\,\mathbf{a}_3) = \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

(b) (8 points) Find the QR factorization of A.

Solution.

$$A = QR = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix},$$

where

$$R = \begin{bmatrix} \|\mathbf{a}_1\| & \mathbf{u}_1 \cdot \mathbf{a}_2 & \mathbf{u}_1 \cdot \mathbf{a}_3 \\ 0 & \|\mathbf{a}_2 - \text{proj}_{V_1}\,\mathbf{a}_2\| & \mathbf{u}_2 \cdot \mathbf{a}_3 \\ 0 & 0 & \|\mathbf{a}_3 - \text{proj}_{V_2}\,\mathbf{a}_3\| \end{bmatrix}.$$

3. (a) (8 points) Find the linear function $y = c_0 + c_1 x$ that best fits the following data using least squares.

x	-6	-2	1	7
y	-1	2	1	6

Solution. We must find the least squares solution to

$$A\mathbf{x} = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \mathbf{b}.$$

This is equivalent to solving the normal equations, $A^T A\mathbf{x} = A\mathbf{b}$ or solving

$$\begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 45 \end{bmatrix}.$$

The solution of this system is $c_0 = 2$ and $c_1 = 1/2$. Thus,

$$y = 2 + \frac{1}{2}x.$$

(b) (8 points) Let A be an $m \times n$ matrix such that $\operatorname{rank}(A) = n$ and $\mathbf{b} \in \mathbb{R}^m$. If $A = QR$ is the QR factorization of A, show that the unique least squares solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$\widehat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}.$$

Solution. Since rank $A = n$, the solution to the normal equations is

$$\begin{aligned} \widehat{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= [(QR)^T(QR)]^{-1}(QR)^T\mathbf{b} \\ &= [R^TQ^TQR]^{-1}R^TQ^T\mathbf{b} \\ &= [R^TR]^{-1}R^TQ^T\mathbf{b} \\ &= R^{-1}(R^T)^{-1}R^TQ^T\mathbf{b} \\ &= R^{-1}Q^T\mathbf{b}. \end{aligned}$$

4. Let $\mathbf{u}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ be a basis for $\mathbb{R}^2$.

(a) (8 points) If $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find c_1 and c_2 such that $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$.

Solution. Since

$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we know that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = -4\begin{bmatrix} 5 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

(b) (8 points) If $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is another basis for $\mathbb{R}^2$, find k_1 and k_2 such that $-2\mathbf{u}_1 + 3\mathbf{u}_2 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$.

Solution. The transition matrix that sends coordinates with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}.$$

Thus, the coordinates of $-2\mathbf{u}_1 + 3\mathbf{u}_2$ are

$$\begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

or $-2\mathbf{u}_1 + 3\mathbf{u}_2 = 6\mathbf{v}_1 - 7\mathbf{v}_2$.

5. (8 points) Use row and column operations to find the determinant of the matrix

$$A = \begin{bmatrix} a_1+1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2+1 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3+1 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n+1 \end{bmatrix}$$

where $a_1, \dots a_n$ are real numbers.

Solution. Subtracting the first row of A from each subsequent row, we have

$$\det(A) = \det \begin{bmatrix} a_1+1 & a_2 & a_3 & \dots & a_n \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Now add each column from the first column to get

$$\det(A) = \det \begin{bmatrix} (1+a_1+a_2+a_3+\cdots+a_n) & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Thus, $\det(A) = 1 + a_1 + a_2 + a_3 + \cdots + a_n$.

6. (a) (8 points) Let A be an $n \times n$ skew-symmetric matrix. That is, $A^T = -A$. If n is odd, show that A cannot be invertible. [*Hint*: Show $\det A = 0$.]

   Solution. Since $\det(A) = \det(A^T) = \det(-A) = -\det(A)$, we know that $\det(A) = 0$. Therefore, A cannot be invertible.

   (b) (8 points) Let A be an $n \times n$ skew-symmetric matrix and $\mathbf{x}$ be an $n \times 1$ column vector. Show that $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

   Solution. First notice that $\mathbf{x}^T A\mathbf{x}$ is a scalar. Thus, $\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T$. Consequently,

   $$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x} = -\mathbf{x}^T A\mathbf{x},$$

   and $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

7. (a) (8 points) Let V be the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}.$$

Find a basis for $V^\perp$.

Solution. The subspace V is image of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}.$$

Since $V^\perp = \text{image}(A)^\perp = \ker(A^T)$, we must find the kernel of

$$A^T = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{bmatrix}.$$

However, this is just the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (8 points) Let A be an $m \times n$ matrix. Show that $\ker(A^TA) = \ker(A)$. [*Hint*: First show that if $\mathbf{x} \in \ker(A^TA)$, then $A\mathbf{x}$ is in both image(A) and $\ker(A^T)$.]

Solution. It is easy to show that $\ker(A) \subset \ker(A^TA)$. If $\mathbf{x} \in \ker A$, then $A\mathbf{x} = \mathbf{0}$. Thus, $A^TA\mathbf{x} = \mathbf{0}$, and $\mathbf{x} \in \ker(A^TA)$.
To show that $\ker(A^TA) \subset \ker(A)$, let $\mathbf{x} \in \ker(A^TA)$. Then $A^TA\mathbf{x} = \mathbf{0}$. Thus, $A\mathbf{x} \in \ker(A^T)$. Since $A\mathbf{x} \in \text{image}(A)$ and $\ker(A^T) = \text{image}(A)^\perp$, we know that

$$A\mathbf{x} \in \ker(A^T) \cap \text{image}(A) = \{\mathbf{0}\}.$$

Therefore, $\mathbf{x} \in \ker(A)$.