(1) Notice that $T(x,y) = xy$ satisfies Laplace's Equation, since $\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} = 0$. So if the center of the disk is (x_0, y_0) the temperature there could be $x_0 y_0$.

Can we come up with another solution? Let's assume the disc is located in the square $[0,\pi] \times [0,\pi]$, and let the temperature function on the disk be $s(x,y)$, so that $s(x,y) = xy$ on the boundary and $\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0$. Then $T(x,y) = s(x,y) - xy$ gives the

temperature of a disc s.t. the temperature is 0 on the boundary. Extend $T(x,y)$ to be 0 on the points in $[0,\pi] \times [0,\pi]$ not in the disk. This extended function satisfies Laplace's equation as well as the boundary conditions $T(x,0) = T(x,\pi) = T(0,y) = T(\pi,y) = 0$. (We can do this because T is continuous, being 0 on the boundary of the disk.) Given these boundary conditions, our methods only come up with 1 solution for $T(x,y)$: $T(x,y) = 0$.

Since $T(x,y) = s(x,y) - xy$, we weren't able to come up with $s(x,y) \neq xy$. This doesn't mean no such $s(x,y)$ exists, but it makes $x_0 y_0$ a very acceptable answer.

(2) If $s(x,y)$ satisfies Laplace's equation with boundary conditions

$$s(0,y) = 0 \qquad s(\pi,y) = \begin{cases} y & y \le \pi/2, \\ \pi - y & y \ge \pi/2 \end{cases}$$
$$s(x,0) = 0 \qquad s(x,\pi) = 0$$

and $h(x,y)$ satisfies Laplace's equation with boundary conditions

$$h(0,y) = 0 \qquad h(\pi,y) = 0$$
$$h(x,0) = 0 \qquad h(x,\pi) = \begin{cases} x & x \le \pi/2, \\ \pi - x & x \ge \pi/2 \end{cases}$$

then $T(x,y) = s(x,y) + h(x,y)$ satisfies Laplace's equation with

$$T(0,y) = 0 \qquad T(\pi,y) = \begin{cases} y & y \le \pi/2, \\ \pi - y & y \ge \pi/2 \end{cases}$$
$$T(x,0) = 0 \qquad T(x,\pi) = \begin{cases} x & x \le \pi/2, \\ \pi - x & x \ge \pi/2 \end{cases}$$

How do we find s and h?

From page 20, we know that

$$s(x,y) = \sum_{n=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \frac{\sinh(2m+1)x}{\sinh(2m+1)\pi} \sin(2m+1)y$$

Just by switching x and y we obtain

$$h(x,y) = \sum_{m=0} \frac{4(-1)^m}{\pi(2m+1)^2} \frac{\sinh(2m+1)y}{\sinh(2m+1)\pi} \sin(2m+1)x$$

So $T(x,y) = s(x,y) + h(x,y)$

$$= \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2 \sinh(2m+1)\pi} \left(\sinh(2m+1)x \sin(2m+1)y + \sinh(2m+1)y \sin(2m+1)x \right)$$

③ We know that

$$u(x,t) = \sum_{n=0}^{\infty} (a_n \sin nt + b_n \cos nt) \sin nx$$

is a general solution to $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $u(0,t) = u(\pi,t) = 0$.

If the string is initially undisturbed, then $u(x,0) = 0$.

Since $u(x,0) = \sum_{n=0}^{\infty} b_n \sin nx = 0$, we have $b_n = 0$ for all n.

If $\frac{\partial u}{\partial t}(x,0) = \begin{cases} x & x \le \pi/2 \\ \pi - x & x \ge \pi/2 \end{cases} = \Theta(x)$

then $\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} n a_n \cos nt \sin nx \Big|_{t=0} = \sum_{n=0}^{\infty} n a_n \sin nx = \Theta(x)$.

We need to find a Fourier sine series for $\Theta(x)$. Extend it over the interval $[-\pi, \pi]$: $\Theta(x) = \begin{cases} x & -\pi/2 \le x \le 0 \\ -\pi - x & -\pi \le x \le -\pi/2 \end{cases}$

Since $\Theta(x) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \sin(2m+1)x$, we obtain

$$(2m+1)\, a_{2m+1} = \frac{4(-1)^m}{\pi(2m+1)^2}$$

$$\text{so} \quad a_{2m+1} = \frac{4(-1)^m}{\pi(2m+1)^3}, \quad a_{2m} = 0$$

$$\text{and} \quad u(x,t) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^3} \sin(2m+1)t \sin(2m+1)x$$

⑤ Let $T(x,y) = u(x)\, v(y)$.

$$\text{Then} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = u''(x)\,v(y) + u(x)\,v''(y) = 0$$

$$\text{so} \quad \frac{u''(x)}{u(x)} = -\frac{v''(y)}{v(y)} = \text{constant} = \pm\lambda^2$$

There are 3 possibilities:

(I) $u(x) = Ae^{\lambda x} + Be^{-\lambda x}$, $v(y) = C\cos\lambda y + D\sin\lambda y$ $(\lambda \neq 0)$

(II) $u(x) = Ax + B$, $v(y) = Cy + D$ $(\lambda = 0)$

(III) $u(x) = A\cos\lambda x + B\sin\lambda x$, $v(y) = Ce^{\lambda y} + De^{-\lambda y}$ $(\lambda \neq 0)$

We have boundary conditions

$$\frac{\partial T}{\partial x}(0,y) = u'(0)\,v(y) = 0 \Rightarrow u'(0) = 0$$

$$\frac{\partial T}{\partial y}(x,0) = u(x)\,v'(0) = 0 \Rightarrow v'(0) = 0$$

$$\frac{\partial T}{\partial y}(x,\pi) = u(x)\,v'(\pi) = 0 \Rightarrow v'(\pi) = 0.$$

$$\cancel{\frac{\partial T}{\partial x}(\pi,y) = u'(\pi)\,v(y) = \sin 2y \Rightarrow u'(\pi) \text{ is a constant}}$$

~~⇒ v(y) is a constant multiple of sin 2y.~~

In case I, we have

$$u'(0) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}\Big|_{x=0} = A\lambda - B\lambda = 0 \Rightarrow A = B$$

$$v'(0) = -C\lambda \sin\lambda y + D\lambda\cos\lambda y \Big|_{y=0} = D\lambda = 0 \Rightarrow D = 0.$$

$$v'(\pi) = -C\lambda \sin\lambda\pi = 0 \Rightarrow C = 0 \text{ or } \sin\lambda\pi = 0$$

If $C=0$, $v(y)=0$ and $T(x,y)=0$, which is uninteresting.
If $\sin\lambda\pi = 0$, then λ is an integer.

So $u(x) = A\cosh\lambda x$, $v(y) = C\cos\lambda y$, $T(x,y) = AC\cosh\lambda x\cos\lambda y$, λ int.

In case II, we have

$u'(0) = A = 0$, $v'(0) = C = 0$, $u(x) = B$, $v(y) = D$, $T(x,y) = BD$.
This corresponds to $\lambda = 0$ in case I.

In case III, we have

$$u'(0) = -A\lambda \sin\lambda \cancel{x} + B\lambda\cos\lambda x\Big|_{x=0} = B\lambda = 0 \Rightarrow B = 0$$

$$v'(0) = C\lambda e^{\lambda y} - D\lambda e^{-\lambda y}\Big|_{y=0} = C\lambda - D\lambda = 0 \Rightarrow C = D.$$

$$v'(\pi) = C\lambda e^{\lambda y} - C\lambda e^{-\lambda y}\Big|_{y=\pi} = C\lambda(e^{\pi\lambda} - e^{-\pi\lambda}) = 0$$

$$\Rightarrow C = 0.$$

So this case corresponds to $T(x,y)=0$, and is therefore uninteresting.

We can consider a general solution to be of the form

$$\sum_{n=\cancel{0}\infty}^{\infty} c_n \cosh nx \cos ny$$

Since $\cosh nx = \cosh(-nx)$, $\cos ny = \cos(-ny)$, we can use

$$\sum_{n=0}^{\infty} c_n \cosh nx \cos ny.$$

Now let's look at the fourth boundary condition:

$$\frac{\partial T}{\partial x}(\pi, y) = \sin 2y$$

or $$\sum_{n=0}^{\infty} n c_n \sinh n\pi \cos ny = \sin 2y.$$

If we could find a Fourier cosine series for $\sin 2y$,

$$\sin 2y = \sum_{n=0}^{\infty} b_n \cos ny$$

then setting $c_n = \dfrac{b_n}{\sinh n\pi}$ would finish the problem.

Let $\Theta(y) = \begin{cases} \sin 2y & 0 \le y \le \pi \\ -\sin 2y & -\pi \le y \le 0 \end{cases}$

$\Theta(y)$ is an even function, so it has a Fourier cosine series:

$$b_0 = \left\langle \Theta(y), \frac{1}{\sqrt{2}} \right\rangle / \sqrt{2} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \Theta(y) \cdot \frac{1}{\sqrt{2}} \, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta(y) \, dy = \frac{1}{\pi} \int_0^{\pi} \sin 2y \, dy = \frac{1}{\pi}\left(-\frac{\cos 2y}{2}\right)\Big|_0^{\pi} = 0$$

For $n > 0$

$$b_n = \langle \Theta(y), \cos ny \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \Theta(y) \cos ny \, dy = \frac{2}{\pi} \int_0^{\pi} \sin 2y \cos ny \, dy$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+2)y - \sin(n-2)y \, dy$$

For $n \neq 2$

$$\int_0^{\pi} \sin(n+2)y - \sin(n-2)y \; dy = -\frac{\cos(n+2)y}{n+2} + \frac{\cos(n-2)y}{n-2}\Bigg|_0^{\pi}$$

$$= -\frac{(-1)^{n+2}-1}{n+2} + \frac{(-1)^{n-2}-1}{n-2} = \left[(-1)^n - 1\right]\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

If n is even, this is 0; if n is odd, the integral evalutes to $\frac{-8}{n^2-4}$.

For $n=2$

$$\int_0^{\pi} \sin(n+2)y - \sin(n-2)y \, dy = \int_0^{\pi} \sin 4y \, dy = 0.$$

So $b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-8}{\pi(n^2-4)} & \text{if } n \text{ is odd.} \end{cases}$

Setting $c_n = \dfrac{b_n}{\sinh n\pi}$, we obtain

$$T(x,y) = \sum_{n=0}^{\infty} c_n \cosh nx \cos ny$$

$$c_n = \begin{cases} 0 & \text{if } n \text{ even} \\ -\dfrac{8}{\pi(n^2-4)\sinh n\pi} & \text{if } n \text{ odd.} \end{cases}$$

or $$T(x,y) = \sum_{m=0}^{\infty} \frac{-8}{\pi\left((2m+1)^2-3\right)\sinh (2m+1)\pi} \cosh(2m+1)x \cos(2m+1)y$$