Gradients & Level Surfaces

Recall from Monday that the *gradient* of a function f is the vector ∇f defined by

$$\nabla f(x,y) = \text{grad}\, f = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

in the plane or

$$\nabla f(x,y,z) = \text{grad}\, f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

in three-space.

1 For each of the following functions $f(x,y)$ in the plane, do the following:

(i) Compute the gradient ∇f (or grad f).

(ii) Identify the level curve $f(x,y) =$ constant through the point $(x,y) = (1,1)$.

(iii) Find a parameterization $\mathbf{r}(t)$ of this curve.

(iv) Verify that the tangent vector $\mathbf{r}'(t)$ and the gradient vector ∇f are perpendicular at the point $(x,y) = (1,1)$.

(a) $f(x,y) = 3x - y$

(b) $f(x,y) = 2x^2 + 3y^2$

2 For each of the following functions $F(x,y,z)$ in the plane, do the following:

(i) Compute the gradient ∇F (or grad F).

(ii) Identify the level surface $F(x,y,z) =$ constant through the point $(x,y,z) = (1,1,1)$.

(iii) Find the tangent plane to the level surface from part (ii). Recall that the tangent plane to a surface $z = f(x,y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

(iv) Verify that the gradient vector ∇F is perpendicular to (tangent plane of) the level surface at the point $(x,y,z) = (1,1,1)$.

(a) $F(x,y,z) = 3x + 2y + z$

(b) $F(x,y,z) = x^2 + y^2 - z^2$

These problems illustrate the following point:

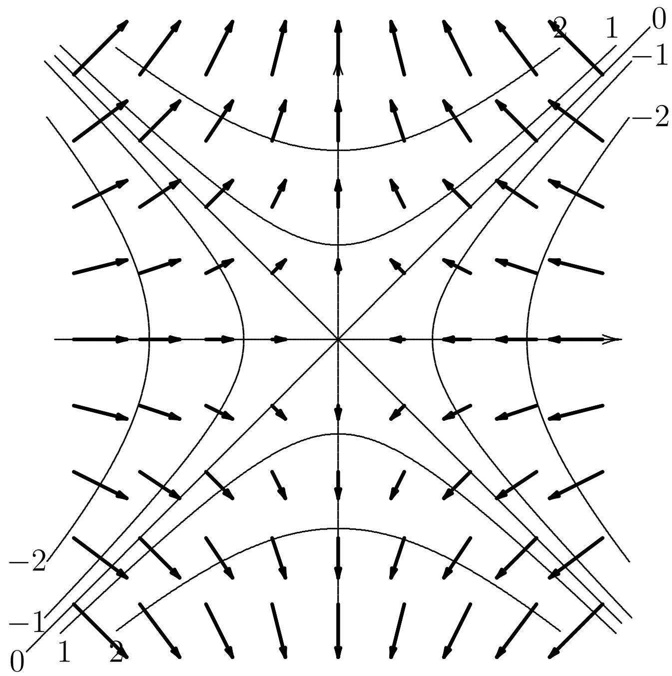
> The gradient of a function is always perpendicular to the level curves (or level surfaces) of the function!

Gradients & Level Surfaces

There are two important facts about the gradient vector:

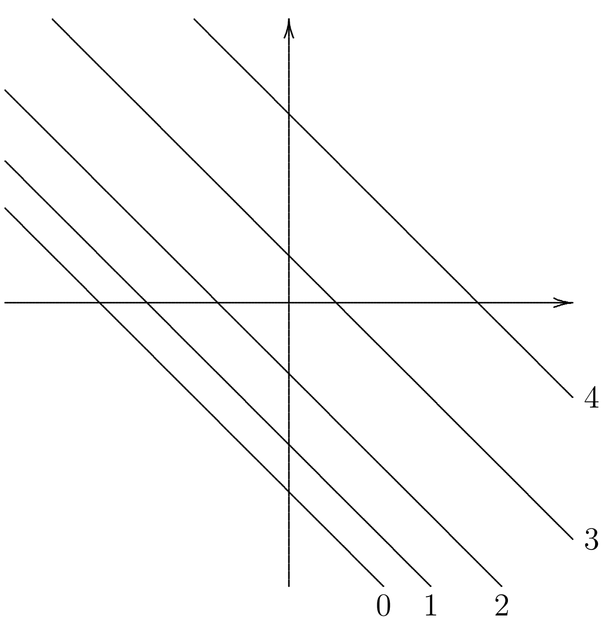
- grad f (or ∇f) is perpendicular to the level curves of f (as we saw on page one of this handout)
- $|\text{grad}\, f|$ (or the magnitude of ∇f) is the rate of change of f in the direction of grad f

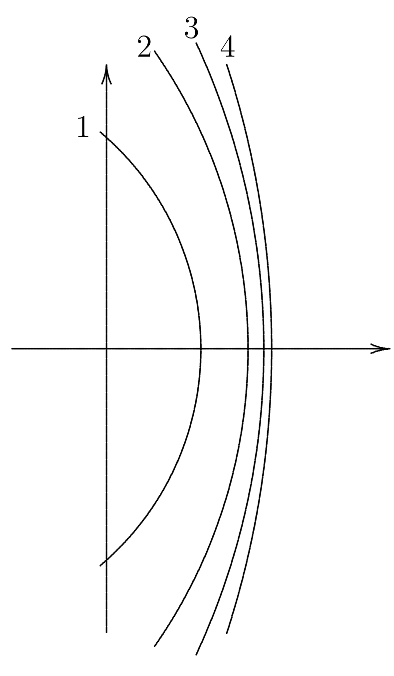
Here is an example sketch of the level curves of $f(x, y) = y^2 - x^2$ and the associated gradient vector field:



(The arrows shown here are in fact one-tenth the *actual* length of the gradient, but they're shrunk to make the picture cleaner. Here $\nabla f = \langle f_x, f_y \rangle = \langle -2x, 2y \rangle$, so $|\nabla f| = \sqrt{4x^2 + 4y^2} = 2r$.)

Use the two facts shown above to sketch the gradient vector field given the following contour plots (pictures of level curves):





Gradients & Level Surfaces – Answers / Solutions

1 (a) (i) $\nabla f = \langle 3, -1\rangle$ at every point, not just $(x, y) = (1, 1)$.

(ii) The curve $f(x, y) = f(1, 1)$ is $3x - y = 2$, a line.

(iii) A simple parameterization is $x = t$, so $y = 3t - 2$, or $\mathbf{r}(t) = \langle t, 3t - 2\rangle$.

(iv) The tangent vector in our parameterization is always $\mathbf{r}'(t) = \langle 1, 3\rangle$, so $\nabla f \cdot \mathbf{r}'(t) = 0$ for all t, not just for $t = 1$ (the point $(x, y) = (1, 1)$).

(b) (i) $\nabla f = \langle 4x, 6y\rangle$, so $\nabla f(1, 1) = \langle 4, 6\rangle$.

(ii) The curve $f(x, y) = f(1, 1)$ is $2x^2 + 3y^2 = 5$, an ellipse.

(iii) One parameterization is

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \left\langle \sqrt{\tfrac{5}{2}}\cos(t), \sqrt{\tfrac{5}{3}}\sin(t)\right\rangle.$$

I found this by using the parameterization $\langle u, v\rangle = \langle \sqrt{5}\cos(t), \sqrt{5}\sin(t)\rangle$ for the circle $u^2 + v^2 = 5$, then writing our ellipse as $(\sqrt{2}x)^2 + (\sqrt{3}y)^2 = 5$ and making the substitutions $u = \sqrt{2}x$ and $v = \sqrt{3}y$.

(iv) The tangent vector in our parameterization is $\mathbf{r}'(t) = \left\langle -\sqrt{\tfrac{5}{2}}\sin(t), \sqrt{\tfrac{5}{3}}\sin(t)\right\rangle$. You might think we need to find t_0 when $(x, y) = (1, 1)$, but in reality we only need to find $\cos(t_0)$ and $\sin(t_0)$. We know that $\mathbf{r}(t_0) = \langle 1, 1\rangle$, so

$$\left\langle \sqrt{\frac{5}{2}}\cos(t_0), \sqrt{\frac{5}{3}}\sin(t_0)\right\rangle = \langle 1, 1\rangle \qquad \text{or} \qquad \cos(t_0) = \sqrt{\frac{2}{5}} \quad \text{and} \quad \sin(t_0) = \sqrt{\frac{3}{5}}.$$

Thus $\mathbf{r}'(t_0) = \left\langle -\sqrt{\tfrac{5}{2}}\cdot\sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{5}{3}}\cdot\sqrt{\tfrac{2}{5}}\right\rangle = \left\langle -\sqrt{3/2}, \sqrt{2/3}\right\rangle$. Thus

$$\nabla f(1, 1) \cdot \mathbf{r}'(t_0) = \langle 4, 6\rangle \cdot \left\langle -\sqrt{3/2}, \sqrt{2/3}\right\rangle = -4\sqrt{3/2} + 6\sqrt{2/3} = 0.$$

Thus the two vectors are perpendicular.

2 (a) (i) $\nabla F = \langle 3, 2, 1\rangle$

(ii) The level surface $F(x, y, z) = F(1, 1, 1)$ is the plane $3x + 2y + z = 6$.

(iii) To use this formula, we solve for z: $z = f(x, y) = 6 - 3x - 2y$. Thus the tangent line is

$$z - 1 = -3(x - 1) - 2(y - 1) \qquad \text{or} \qquad 3x + 2y + z = 6.$$

Fancy that! The tangent plane to a plane is the plane itself!

(iv) The gradient $\nabla F = \langle 3, 2, 1\rangle$ is the same as the normal to the tangent plane (and the level surface itself); hence the gradient is perpendicular to the tangent plane of the level surface.

(b) (i) $\nabla F = \langle 2x, 2y, -2z\rangle$, so $\nabla F(1,1,1) = \langle 2, 2, -2\rangle$.

(ii) The level surface $F(x, y, z) = F(1, 1, 1)$ is the one-sheeted hyperboloid $x^2+y^2-z^2 = 1$.

(iii) To use this formula, we solve for z: $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$ (we want the positive square root since $z = 1$ at our point). At $(x, y) = (1, 1)$, the derivative $f_x(1, 1)$ is easy to find:

$$f_x = \frac{1}{2}\left(x^2 + y^2 - 1\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 - 1}},$$

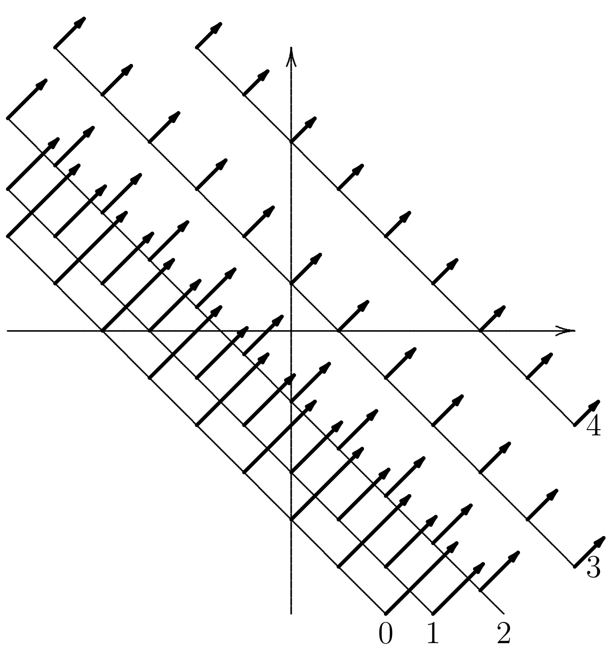
so $f_x(1, 1) = \frac{1}{\sqrt{1^2+1^2-1}} = 1$ at the point $(x, y) = (1, 1)$. Thus the tangent line is

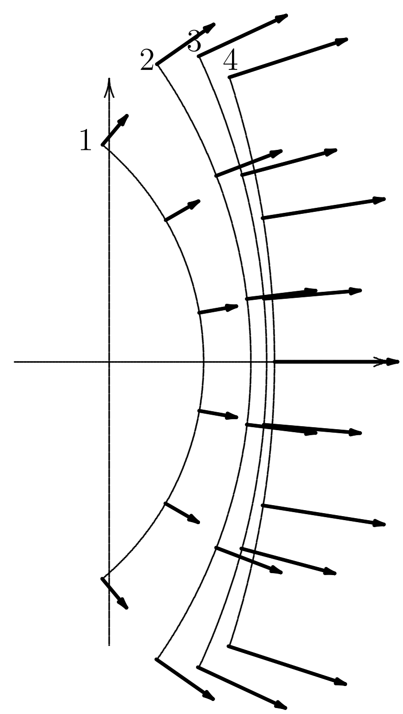
$$z - 1 = 1(x - 1) + 1(y - 1) \qquad \text{or} \qquad x + y - z = 1.$$

Note that the normal to this tangent plane is $\mathbf{n} = \langle 1, 1, -1\rangle$.

(iv) The gradient $\nabla F(1, 1, 1) = \langle 2, 2, -2\rangle$ is parallel to the normal $\mathbf{n} = \langle 1, 1, -1\rangle$ to the tangent plane. As before, therefore, the gradient is perpendicular to the tangent plane of the level surface.

Back Page: Here are the two graphs with some gradient vectors drawn in:





In both cases the gradient vectors have been scaled to make sure the picture is not overwhelmed (or underwhelmed) with arrows.