

## INTEGRAL THEOREM PROBLEMS

Math21a, O. Knill

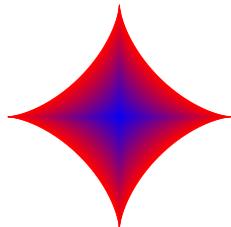
SUMMARY. This is a collection of problems on line integrals, Green's theorem, Stokes theorem and the divergence theorem. Some of them are more challenging.

### LINE INTEGRALS GREEN THEOREM.

The curve  $\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$  is called a **hypocycloid**. It bounds a region  $R$  in the plane.

- Calculate the line integral of the vector field  $F(x, y) = \langle x, y \rangle$  along the curve.
- Find the area of the hypocycloid.

- Because  $\text{curl}(\vec{F}) = 0$  the result is zero by Green's theorem.
- Use the vector field  $\vec{F}(x, y) = \langle 0, x \rangle$  which has  $\text{curl}(\vec{F}) = 1$ . The line integral is  $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^{2\pi} \cos^3(t) 3 \sin^2(t) \cos(t) dt = \int_0^{2\pi} 3 \cos^4(t) \sin^2(t) dt = 3\pi/8$ . (To compute the integral, use that  $8 \cos^4(t) \sin^2(t) = \cos(2t) \sin^2(2t) + \sin^2(2t)$ ).



### LENGTH OF CURVE AND LINE INTEGRALS.

Assume  $C : t \mapsto \vec{r}(t)$  is a closed path in space and  $\vec{F}(\vec{r}(t))$  is the unit tangent vector to the curve (that is a vector parallel to the velocity vector which has length 1).

- What is  $\int_C \vec{F} dr$ ?
- Can  $\vec{F}$  be a gradient field?

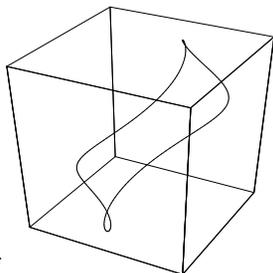
Answer:

- $\vec{F}(\vec{r}(t)) = \vec{r}'(t)/|\vec{r}'(t)|$ . By definition of the line integral,

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \cdot \vec{r}'(t) dt = \int_a^b |\vec{r}'(t)| dt,$$

which is the length of the curve.

- No: If  $\vec{F}$  were a gradient field, then by the fundamental theorem of line integrals, we would have that the line integral along a closed curve is zero. But because this is the length of the curve, this is not possible.



### SURFACE AREA AND FLUX.

Assume  $S : (u, v) \mapsto \vec{r}(u, v)$  is a closed surface in space and  $\vec{F}(\vec{r}(u, v))$  is the unit normal vector on  $S$  (which points in the direction of  $\vec{r}_u \times \vec{r}_v$ ).

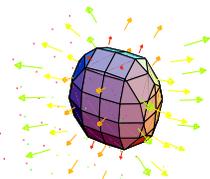
- What is  $\int_S \vec{F} \cdot d\vec{S}$ ?
- Is it possible that  $\vec{F}$  is the curl of another vector field  $\vec{G}$ ?
- Is it possible that  $\text{div}(\vec{F}) = \langle 0, 0, 0 \rangle$  everywhere inside the surface.

Answer:

- $\vec{F}(\vec{r}(u, v)) = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$ . By definition of the flux integral,  $\int_S \vec{F} \cdot d\vec{S} = \int_R \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v = \int_R (\vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|) \cdot \vec{r}_u \times \vec{r}_v = \int_R |\vec{r}_u \times \vec{r}_v| dudv$  which is the area of the surface.

- No, if  $\vec{F}$  were the curl of another field  $G$ , then the flux of  $\vec{F}$  through the closed surface would be zero. But since it is the area, this is not possible.

- From the divergence theorem follows that  $\text{div}(\vec{F})$  is nonzero somewhere inside the surface.

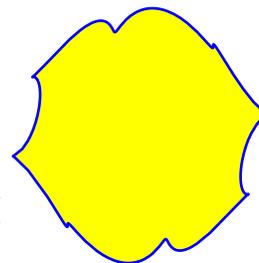


### GREENS THEOREM AND LAPLACIAN.

Assume  $R$  is a region in the plane and let  $\vec{n}$  denote the unit normal vector to the boundary  $C$  of  $R$ . For any function  $u(x, y)$ , we use the notation  $\partial f / \partial u = \text{grad}(u) \cdot \vec{n}$  which is the directional derivative of  $u$  into the direction  $\vec{n}$  normal to  $C$ . We also use the notation  $\Delta u = u_{xx} + u_{yy}$ . Show that

$$\int_C \partial u / \partial n \cdot dr = \iint_R \Delta u dA$$

Answer: Define  $\vec{F}(x, y) = \langle -B, A \rangle$  if  $\partial u / \partial n = \langle A, B \rangle$ . The left integral is the line integral of  $\vec{F}$  along  $C$ . The right integral is the double integral over  $\Delta u = \text{curl}(\vec{F})$ .



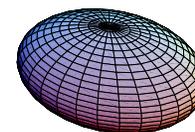
### STOKES THEOREM OR DIVERGENCE THEOREM

Find  $\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ , where  $S$  is the ellipsoid  $x^2 + y^2 + 2z^2 = 10$  and  $\vec{F}(x, y, z) = \langle \sin(xy), e^x, -yz \rangle$ .

Answer. The integral is zero because the boundary of  $S$  is empty. This fact can be seen using Stokes theorem. It can also be seen by divergence theorem

$$\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iiint_V \text{div}(\text{curl}(\vec{F})) dV.$$

using  $\text{div}(\text{curl})(\vec{F}) = 0$ .

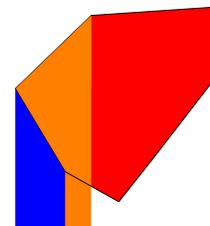


### AREA OF POLYGONS.

If  $P_i = (x_i, y_i), i = 1, \dots, n$  are the edges of a polygon in the plane, then its area is  $A = \sum_i (x_i - x_{i+1})(y_{i+1} + y_i)/2$ .

The proof is an application of Green's theorem. The line integral of the vector field  $F(x, y) = \langle -y, 0 \rangle$  through the side  $P_i, P_{i+1}$  is  $(x_i - x_{i+1})(y_{i+1} + y_i)/2$ , because  $(x_{i+1} - x_i)$  is the projected area onto the x-axis and  $(y_{i+1} + y_i)/2$  is the average value of the vector field on that side. Because  $\text{curl}(F)(x, y) = 1$  for all  $(x, y)$ , the result follows from Greens theorem.

The result can also be seen geometrically:  $(x_i - x_{i+1})(y_{i+1} + y_i)/2$  is the signed area of the trapezoid  $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, y_{i+1})$ . In the picture, we see two of them. The second one is taken negatively.

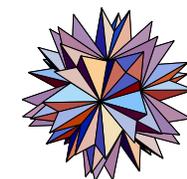


### VOLUME OF POLYHEDRA.

Verify with the divergence theorem: If  $P_i = (x_i, y_i, z_i)$  are the edges of a polyhedron in space and  $F_j = \{P_{i_1}, \dots, P_{i_{k_j}}\}$  are the faces, then  $V = \sum_j A_j \bar{z}_j$  where  $A_j$  is the area of the  $xy$ -projection (\*) of the polygon  $F_j$  and  $\bar{z}_j = (z_{i_1} + \dots + z_{i_{k_j}})/k_j$  is the average  $z$  value of the face  $F_j$ .

Solution. The vector field  $\vec{F}(x, y, z) = z$  has divergence 1. The flux through a face  $F$  is  $|F_j|(z_{i_1} + \dots + z_{i_{k_j}})/k_j$ . Gauss theorem assures that the volume is the sum of the fluxes  $A_j \bar{z}_j$  through the faces.

(\*) The projection of a polygon is the "shadow" when projecting from space along the  $z$ -axis onto the  $xy$ -plane. A triangle  $(1, 0, 1), (1, 1, 0), (0, 1, 2)$  for example would be projected to the triangle  $(1, 0), (1, 1), (0, 1)$ .



### STOKES AND GAUSS TOGETHER.

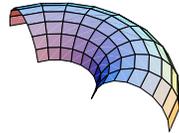
Can you derive  $\text{div}(\text{curl}(\vec{F})) = 0$  using Gauss and Stokes theorem?  
 Consider a sphere  $S$  of radius  $r$  around a point  $(x, y, z)$ . It bounds a ball  $G$ . Consider a vector field  $\vec{F}$ . The flux of  $\text{curl}(\vec{F})$  through  $S$  is zero because of Stokes theorem. Gauss theorem tells that the integral of  $f = \text{div}(\text{curl}(\vec{F}))$  over  $G$  is zero. Because  $S$  was arbitrary,  $f$  must vanish everywhere.



### FUNDAMENTAL THEOREM AND STOKES.

Can you derive the identity  $\text{curl}(\text{grad}(\vec{F})) = 0$  from integral theorems?

To see that the vector field  $\vec{G} = \text{curl}(\text{grad}(\vec{F})) = 0$  is identically zero, it is enough to show that the flux of  $\vec{G}$  through any surface  $S$  is zero. By Stokes theorem, the flux through  $S$  is  $\int_C \text{grad}(\vec{F}) \cdot d\vec{r}$ . By the fundamental theorem of line integrals, this is zero.



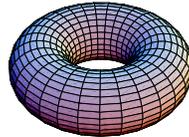
### VOLUME COMPUTATION WITH GAUSS.

Calculate the volume of the torus  $T(a, b)$  enclosed by the surface  $\vec{r}(u, v) = ((a + b \cos(v)) \cos(u), (a + b \cos(v)) \sin(u), b \sin(v))$  using Gauss theorem and the vector field  $\vec{F}(x, y, z) = \langle x, y, 0 \rangle / 2$ .

The vector field  $\vec{F}$  has divergence 1. The parameterization of the torus gives

$$\vec{r}_u \times \vec{r}_v = b(a + b \cos(v)) \langle \cos(u) \cos(v), \cos(v) \sin(u), \sin(v) \rangle.$$

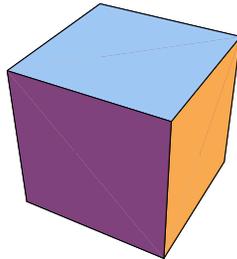
The flux of this vectorfield through the boundary of the torus is  $\int_0^{2\pi} \int_0^{2\pi} b(a + b \cos(v))^2 \cos(v) \, dudv = 2\pi^2 ab^2$ .



### GAUSS OR STOKES?

You know that the flux of the vector field  $\vec{G} = \text{curl}(\vec{F})(x, y, z)$  through 5 faces of a cube  $D$  is equal to 1 each. What is the flux of the same vector field  $\vec{G}$  through the 6'th face?

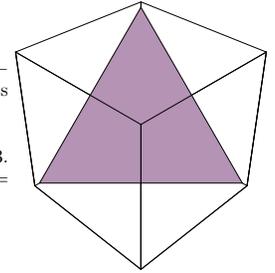
Solution: the problem is best solved with the divergence theorem: because the flux of  $\vec{G}$  through the entire surface is zero, the flux through the 6'th face must cancel the sum of the fluxes 5 through the other 5 surfaces. The result is  $-5$ .



### WORK COMPUTATION USING STOKES.

Calculate the work of the vector field  $\vec{F}(x, y, z) = \langle x - y + z, y - z + x, z - x + y \rangle$ , along the path  $C$  which connects the points  $(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0)$  in that order.

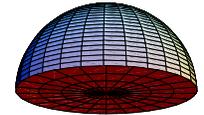
Answer. The line integral over each part is each 1. The total is 3.  $\text{curl}(\vec{F}) = \langle 2, 2, 2 \rangle$  and  $S : (u, v) \mapsto \vec{r}(u, v) = \langle u, v, 1 - u - v \rangle$   $\vec{r}_u \times \vec{r}_v = \langle 1, 1, 1 \rangle$   $\int_S \text{curl}(\vec{F}) \cdot \vec{r}_u \times \vec{r}_v \, dudv = 6$  area of  $S = 3$ .



### STOKES OR GAUSS?

Compute the flux of the vector field  $\vec{F}(x, y, z) = \langle x - x \sin(\sin(z)), 2y, 3z + \sin(\sin(z)) \rangle$  through the upper hemisphere  $S = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0 \}$ .

Answer. We use Gauss:  $\text{div}(\vec{F}) = 6$  and  $\int \int \int_B \text{div}(\vec{F}) \, dV = 6 \text{Vol}(B) = 4\pi$ . We can not easily compute the flux through the hemisphere. However, we can see that the flux through the floor of the region is zero because the normal component  $P$  of of the vector field  $\vec{F} = \langle M, N, P \rangle$  is zero on  $z = 0$ . So: the result is  $4\pi - 0 = 4\pi$ .

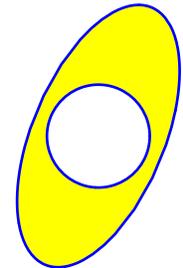


### GREENS THEOREM.

Calculate the work of the vector field  $\vec{F}(x, y) = \frac{1}{x^2 + y^2} \langle -y, x \rangle$  along the boundary of the ellipse  $\vec{r}(t) = \langle 3 \cos(t) + \sin(t), 5 \sin(t) + \cos(t) \rangle$ .

Solution. Take an other curve  $C : x^2 + y^2 \leq 4$  and apply Green's theorem to the region  $R$  bounded by the ellipse and the circle. Because  $\text{curl}(\vec{F})$  is zero in  $D$ , the line integral along the ellipse is the same as the line integral along the circle:  $t \mapsto \vec{r}(t) = \langle 2 \cos(t), 2 \sin(t) \rangle$  with velocity  $\vec{r}'(t) = \langle -2 \sin(t), 2 \cos(t) \rangle$ :

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \frac{\langle -2 \cos(t), 2 \sin(t) \rangle}{4} \cdot \langle -2 \sin(t), 2 \cos(t) \rangle dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$



TRUE/FALSE QUESTIONS ON INTEGRAL THEOREMS.

(TF) The flux of the curl of a vector field through the unit sphere is zero.

(TF) The line integral of the curl of a vector field along a closed curve is zero.

(TF) The line integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of how a curve  $C : t \mapsto \vec{r}(t)$  is parametrized.

(TF) The maximal speed of a curve is independent on how the curve is parametrized.

(TF) The flux integral  $\int_S \vec{F} \cdot d\vec{S}$  through a surface is independent on how the surface  $S$  is parametrized.

(TF) The area  $\int_S 1 d\vec{S}$  of a surface is independent on how the surface  $S$  is parametrized.

(TF) The maximal value of  $\vec{r}_u \times \vec{r}_v$  on a surface  $S$  is independent on how the surface is parametrized.

(TF) There exists a vector field in space which has zero divergence, zero curl but is not a constant field.

(TF) There exists a vector field in space which has zero gradient but is not a constant vector field  $\vec{F}(x, y, z) = \langle a, b, c \rangle$ .

(TF) There exists a function in space which has zero Laplacian  $f_{xx} + f_{yy} + f_{zz} = 0$  but which is not constant.

(TF)  $\text{div}(\text{grad}(\vec{F})) = 0$  and  $\text{div}(\text{curl}(\vec{F})) = 0$  and  $\text{curl}(\text{grad}(\vec{F})) = 0$ .

(TF) The line integral of a gradient field along any part of a level curve  $f = \text{const}$  is zero.

(TF) If  $\text{div}(\vec{F}) = 0$ , then the line integral along any closed curve is zero.

(TF) If  $\text{curl}(\vec{F}) = 0$ , then the line integral along any closed curve is zero.

(TF) If  $\text{div}(\vec{F}) = 0$  then the flux integral along any sphere in space is zero.

(TF) If  $\text{curl}(\vec{F}) = 0$  then the flux integral along any sphere in space is zero.