GLOSSARY CHECKLIST **Math21a, Oliver Knill**

Geometry of Space

- coordinates and vectors in the plane and in space
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- dot product $v.w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos(\alpha)$
- cross product, $v.(v \times w) = 0$, $w.(v \times w) = 0$, $|v \times w| = |v||w|\sin(\alpha)$
- triple scalar product $u \cdot (v \times w)$ volume of parallelepiped
- parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$
- scalar projection $\text{comp}_w(v) = v \cdot w/|w|$
- vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- completion of square: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x-2)^2 + y^2 = 5$
- distance $d(P,Q) = |\vec{PQ}| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$

Lines, Planes, Functions

- symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{z-z_0}{c}$
- plane $ax + by + cz = d$
- parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- switch from parametric to implicit descriptions for lines and planes
- domain and range of functions $f(x,y)$
- graph $G = \{(x, y, f(x,y))\}$
- intercepts: intersections of G with coordinate axis
- traces: intersections with coordinate planes
- generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, hyperboloid paraboloid
- plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
- line $\frac{(x-x_0)}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $\vec{v} = (a, b, c)$
- sets $g(x,y,z) = c$ describe surfaces, example graphs $g(x,y,z) = z - f(x,y)$
- linear equation like $2x + 3y + 5z = 7$ defines plane
- quadratic equation like $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
- distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
- distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$
- distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
- finding plane through three points A, B, C: find normal vector $(a, b, c) = \vec{AB} \times \vec{CB}$

Curves

- plane and space curves $\vec{r}(t)$
- velocity $\vec{r}'(t)$, acceleration $\vec{r}''(t)$
- unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$
- $\vec{r}'(t)$ is tangent to the curve
- $\vec{v} = \vec{r}'$ then $\vec{r} = \int_0^t \vec{v}\, dt + \vec{c}$
- $\vec{r}(t) = (f(t)\cos(t), r(t)\sin(t))$ polar curve to polar graph $r = f(\theta)$.
- $\kappa(t) = |\vec{T}'(t)|/|\vec{r}'(t)|$ curvature

Surfaces

- polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$
- cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$
- spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
- $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
- $g(r, \theta, z) = 0$ cylindrical surface, i.e. $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
- $g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
- $f(x, y) = c$ level curves of $f(x, y)$, normal vectors are $\nabla f(x, y)$
- $g(x, y, z) = c$ level surfaces of $g(x, y, z)$, normal vectors are $\nabla f(x, y, z)$
- circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = \langle r\cos t, r\sin t\rangle$.
- ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$
- sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u, v) = (r\cos u\sin v, r\sin u\sin v, r\cos v)$
- ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u, v) = (a\cos u\sin v, b\sin u\sin v, c\cos v)$
- line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
- plane: $ax + by + cz = d$, $\vec{r}(u, v) = \vec{r}_0 + u\vec{v} + v\vec{w}, (a, b, c) = \vec{v} \times \vec{w}$
- surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u, v) = \langle f(v)\cos(u), f(v)\sin(u), v\rangle$
- graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$

Partial Derivatives

- $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
- partial differential equation PDE: $G(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation
- $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- use $L(x, y)$ to estimate $f(x, y)$ near $f(x_0, y_0)$
- tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
- tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
- estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
- $|f(x, y) - L(x, y)|$ in box R around (x_0, y_0) is $\leq M(|x - x_0| + |y - y_0|)^2/2$, where M is the maximal value of $|f_{xx}(x, y)|, |f_{xy}(x, y)|, |f_{yy}(x, y)|$ in R.
- $f(x, y)$ called differentiable if f_x, f_y are continuous
- $f_{xy} = f_{yx}$ Clairot's theorem
- $\vec{r}_u(u, v)$, $\vec{r}_v$ tangent to surface $\vec{r}(u, v)$

Gradient

- $\nabla f(x, y) = (f_x, f_y)$, $\nabla f(x, y, z) = (f_x, f_y, f_z)$, gradient
- $D_v f = \nabla if \cdot v$ directional derivative
- $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ chain rule
- $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ which contains (x_0, y_0, z_0).
- $\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
- $\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$ normal line to surface $f(x, y, z) = c$ at (x_0, y_0, z_0)
- $(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0$ tangent plane at (x_0, y_0, z_0)
- directional derivative is maximal in the $\vec{v} = \nabla f$ direction
- $f(x, y)$ increases, if we walk on the xy-plane in the ∇f direction
- partial derivatives are special directional derivatives
- if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$

Extrema

$\nabla f(x,y) = (0,0)$, critical point or stationary point
$D = f_{xx}f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
$f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0) local maximum
$f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0) local minimum
$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange equations
$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), g(x,y,z) = c$, λ Lagrange equations
second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle**

Double Integrals

$\int\int_R f(x,y)\ dA$ double integral
$\int_a^b \int_c^d f(x,y)\ dydx$ integral over rectangle
$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$ type I region
$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$ type II region
$\int\int_R f(r,\theta) \boxed{r}\ dr d\theta$ polar coordinates
$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv$ surface area
$\int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$ Fubini
$\int\int_R 1\ dxdy$ area of region R
$\int\int_R f(x,y)\ dxdy$ volume of solid bounded by graph(f) xy-plane

Triple Integrals

$\int\int\int_R f(x,y,z)\ dV$ triple integral
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dydx$ integral over rectangular box
$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\ dzdydx$ type I region
$f(r,\theta,z) \boxed{r}\ dzdrd\theta$ cylindrical coordinates
$\int\int\int_R f(\rho,\theta,z) \boxed{\rho^2 \sin(\phi)}\ dzdrd\theta$ spherical coordinates
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
$V = \int\int\int_R 1\ dV$ volume of solid R
$M = \int\int\int_R \rho(x,y,z)\ dV$ mass of solid R with density ρ
$(\int\int\int_R x\ dV/V, \int\int\int_R y\ dV/V, \int\int\int_R z\ dV/V)$ center of mass

Line Integrals

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ vector field in the plane
$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ vector field in space
$\int_C \vec{F} \cdot \vec{dr} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ line integral
Avoid the notation $\int P dx + Q dy + R dz$.
$\vec{F}(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals

FTL: $\vec{F}(x,y) = \nabla f(x,y)$, $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$
Closed loop property $\int_C \vec{F}\ \vec{dr} = 0$, for all closed curves C
Always equivalent are: closed loop property, conservativeness and gradient field
Mixed derivative test $\text{curl}(\vec{F}) \neq 0$ assures $\vec{F}$ is not a gradient field.
In simply connected domains, $\text{curl}(\vec{F}) = 0$ implies conservativeness.

Green's Theorem

$\vec{F}(x,y) = \langle P, Q \rangle$, curl in two dimensions: $\text{curl}(\vec{F}) = Q_x - P_y = \nabla \times F$.
Green's theorem: C boundary of R, then $\int_C \vec{F} \cdot dr = \int\int_R \text{curl}(\vec{F})\ dxdy$
Area computation: Take F with $\text{curl}(F) = N_x - M_y = 1$ like $F = (-y, 0)$ or $F = (0, x)$ or $F = (-y, x)/2$.
Greens theorem is useful to compute difficult line integrals or difficult 2D integrals.

Flux integrals

$\vec{F}(x,y,z)$ vector field, $S = \vec{r}(R)$ parametrized surface
$\vec{r}_u \times \vec{r}_v$ normal vector, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector
$\vec{r}_u \times \vec{r}_v\ dudv = \vec{dS} = \vec{n}\ dS$ normal surface element
$\int\int_S \vec{F}\ \vec{dS} = \int\int_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$ flux integral

Stokes Theorem

$\vec{F}(x,y,z) = \langle P, Q, R \rangle$, $\text{curl}(P,Q,R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \nabla \times F$
Stokes's theorem: C boundary of surface S, then $\int_C \vec{F} \cdot \vec{dr} = \int\int_S \text{curl}(\vec{F}) \cdot dS$
Stokes theorem is useful to compute difficult flux integrals of $\text{curl}(\vec{F})$ or difficult line integrals.

Div Grad Curl

$\nabla = (\partial_x, \partial_y, \partial_z)$, $\text{grad}(F) = \nabla f$, $\text{curl}(F) = \nabla \times F$, $\text{div}(F) = \nabla \cdot F$
$\text{div}(\text{curl}(\vec{F})) = 0$
$\text{curl}(\text{grad}(\vec{F})) = \vec{0}$
$\text{div}(\text{grad}(f)) = \Delta f$ Laplacian.

Divergence Theorem

$\text{div}(P,Q,R) = P_x + Q_y + R_z = \nabla \cdot F$
Divergence theorem: solid E, boundary S then $\int\int_S F \cdot dS = \int\int\int_E \text{div}(F)\ dV$
The divergence theorem is useful to compute difficult flux integrals or difficult 3D integrals.

Some topology

Simply connected region D: can deform any closed curve within D to a point on curve.
Interior of a region D: points in D for which small neighborhood is still in D.
Boundary of a curve: the end points of the curve if they exist.
Boundary of a surface S are curves which bound the surface, points in the surface which correspond to parameters (u,v) which are not in the interior of the parametrization domain.
Boundary of a solid D: the surfaces which bound the solid, points in the solid which are not in the interior of D.
Closed surface: a surface without boundary like for example the sphere.
Closed curve: a curve with no boundary like for example a knot.

Some surface parametrizations

Sphere of radius ρ: $r(u,v) = (\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v))$
Graph of function $f(x,y)$: $f(u,v) = (u, v, f(u,v))$
Graph of function $f(\phi, r)$ in polar: $f(u,v) = (v\cos(u), v\sin(u), f(u,v))$
Plane containing P and vectors $\vec{u}, \vec{v}$: $r(u,v) = P + u\vec{u} + v\vec{v}$
Surface of revolution: distance $g(z)$ of $z - axis : r(u,v) = (g(v)\cos(u), g(v)\sin(u), v)$
Ex: Cylinder: $r(u,v) = (\cos(u), \sin(u), v)$.
Ex: Cone: $r(u,v) = (v\cos(u), v\sin(u), v)$.
Ex: Paraboloid: $r(u,v) = (\sqrt{r}\cos(u), \sqrt{r}\sin(u), v)$.