Name:

MWF 10 Samik Basu
MWF 10 Joachim Krieger
MWF 11 Matt Leingang
MWF 11 Veronique Godin
TTH 10 Oliver Knill
TTH 115 Thomas Lam

- Please mark the box to the left which lists your section and make sure you have written down your name in teh box above.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which the grader can not read will not receive credit.
- Except for the TF and matching problem, show your work.
- No notes, books, calculators, computers, or other electronic aids can be used.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** The vectors $\langle 1, 2, 1 \rangle$ and $\langle 3, 2, -5 \rangle$ are perpendicular.

Solution:
Indeed, the dot product vanishes.

2) T **F** $|\vec{v} \times \vec{w}| = |v||w|\cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$.

Solution:
It is sin not cos.

3) T **F** The vector $\vec{i} \times (\vec{j} \times \vec{k})$ has length 1.

Solution:
It is the zero vector.

4) **T** F The distance between the z-axis and the line $x - 1 = y = 0$ is 1.

Solution:
You can see that geometrically.

5) **T** F If two vectors $\vec{v}$ and $\vec{w}$ are perpendicular, then the lengths of $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$ are the same.

Solution:
Both are the same by Pythagoras

6) T **F** If $\vec{r}_1(t)$ is a parameterization of a curve and $\vec{r}_2(t)$ is a second parameterization of the same curve and $\vec{r}_1(0) = \vec{r}_2(0)$, then the velocity vectors $\vec{r}_1\,'(t)$ and $\vec{r}_2\,'(t)$ are the same.

Solution:
They are parallel, but not necessarily the same.

7) T F There is a surface which has both hyperbola and parabola as traces.

Solution:
The hyperbolic paraboloid has that.

8) T F The parameterization $\vec{r}(\theta,\phi) = \langle 5\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 3\cos(\phi)\rangle$ with $\theta \in [0, 2\pi), \phi \in [0, 2\pi]$ describes an ellipsoid.

Solution:
Indeed, it is a deformed parameterization of a sphere. We have $x^2/25 + y^2/4 + z^2/9 = 1$. The surface is covered twice.

9) T F If the velocity vector $\vec{r}\,'(t)$ is perpendicular to the vector $\vec{r}(t)$, then the parametrized curve $\vec{r}(t)$ is on a sphere.

Solution:
You have proved this in a home work.

10) T F The function $f(x, y) = 2x^2y^2/(x^2 + y^2)$ with $f(0, 0) = 0$ is continuous in the entire plane.

Solution:
Go into polar coordinates gives the function $r^2 \sin(2\theta)$ which goes to zero for $(x, y) \to (0, 0)$.

11) T F The function $f(x, y) = x^{\sin(x)} + \cos(xy^6)$ satisfies the partial differential equation $f_{xyxy}(x, y) = f_{xxyy}(x, y)$ everywhere in the plane.

Solution:
This is consequence of Clairots theorem.

12) T **F** The surface $f(x,y,z) = x^2 + y^2 - z^2 = -1$ is a one-sheeted hyperboloid.

Solution:
$f(x,y,z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

13) T **F** The curvature of the circle $x^2 + y^2 = 4$ is 2.

Solution:
It is 1/2.

14) T **F** The equation $x^2 + y^2/4 = 1$ in space describes an ellipsoid.

Solution:
The equation describes an elliptical cylinder.

15) T **F** For any two vectors $\vec{v}$, $\vec{w}$ we have $\text{proj}_{\vec{v}}(\vec{w}) = \text{proj}_{\vec{w}}(\vec{v})$.

Solution:
The first vector is parallel to $\vec{v}$, the second vector is parallel to $\vec{w}$.

16) **T** F The set of points in space which have distance 1 from a line form a cylinder.

Solution:
Yes, if the the line is the z-axis, then $x^2 + y^2 = 1$ is the equation of the cylinder.

17) T **F** The surface given in spherical coordinates as $\theta = \pi/3$ is a half cone.

Solution:
No it is a half plane

18) T **F** The velocity vector of a parametric curve $\vec{r}(t)$ always has constant length.

Solution:
Obvious.

19) **T** F If f satisfies the PDE $f_{xx} = f_{tt}$, then $g(t,x) = f(t+x, t-x)$ satisfies the PDE $g_{tx} = 0$.

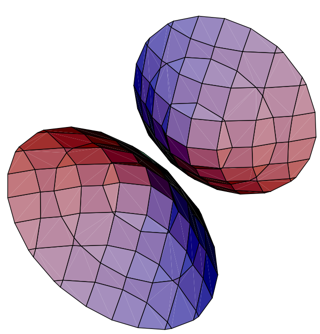
Solution:
Use the chain rule.

20) T **F** The volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u} \times \vec{v}) \times \vec{w}|$.

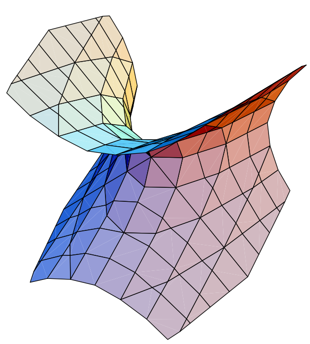
Solution:
The triple scalar product contains also a dot product.

Problem 2) (10 points)

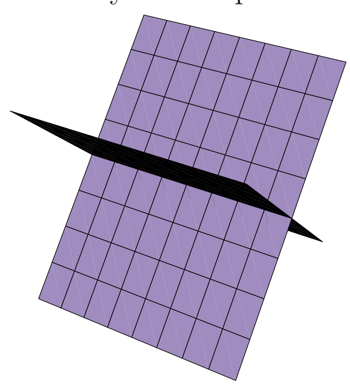
Match the equation with the pictures. No justifications are necessary in this problem.



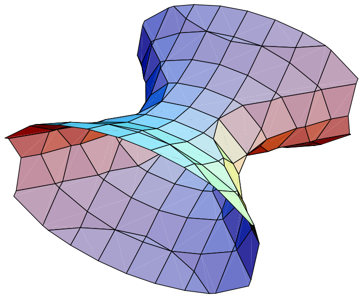
I



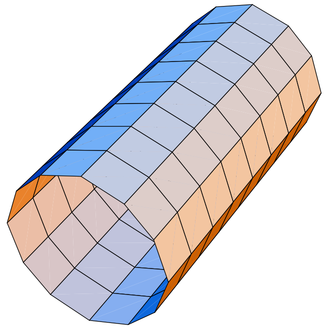
II



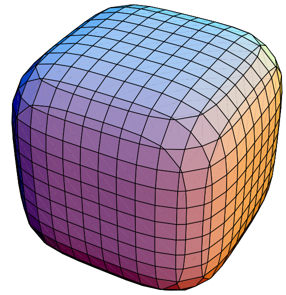
III



IV



V



VI

Enter I,II,III,IV,V,VI here	Equation
	$x^4 + y^4 + z^4 - 1 = 0$
	$-x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + z^2 = 1$
	$-y^2 + z^2 = 0$
	$x^2 - y^2 + 3z^2 - 1 = 0$
	$x^2 - y - z^2 = 0$

Solution:

Enter I,II,III,IV,V,VI here	Equation	Explanation
VI	$x^4 + y^4 + z^4 - 1 = 0$	all three traces are circle like
I	$-x^2 + y^2 - z^2 - 1 = 0$	x-y trace is hyperbola, xz trace is empty
V	$x^2 + z^2 = 1$	no y dependence: cylinder
III	$-y^2 + z^2 = 0$	$y^2 = z^2$ means y=z or y=-z, two planes
IV	$x^2 - y^2 + 3z^2 - 1 = 0$	x z trace is ellipse, xy trace is hyperbola
II	$x^2 - y - z^2 = 0$	parabolas and hyperbola appear as traces

Problem 3) (10 points)

a) (5 points) Find the distance of the point $P = (1, 2, 3)$ to the plane $x + y + z = 1$.

b) (5 points) Find the distance of the point $P = (1, 2, 3)$ to the line $x - 1 = y - 3 = z - 4$.

Solution:
a) The point $Q = (1, 0, 0)$ is on the plane. The vector $\vec{n} = \langle 1, 1, 1 \rangle$ is perpendicular to the surface. Thd distance is

$$d = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|\langle 0, 2, 3 \rangle \cdot \langle 1, 1, 1 \rangle|}{|\sqrt{3}|} = 5/\sqrt{3}$$

The distance is $\boxed{5/\sqrt{3}}$.
b) The symmetric equation stell that $Q = (1, 3, 4)$ is a point on the line and that the vector $\vec{v} = \langle 1, 1, 1 \rangle$ is parallel to the line. With the vector $\vec{PQ} = \langle 0, 1, 1 \rangle$, the distance is

$$d = \frac{\vec{PQ} \times \vec{v}|}{|\vec{v}|} = |\langle 0, 1, -1 \rangle|/\sqrt{3} = \sqrt{2}/\sqrt{3}\ .$$

The distance is $\boxed{\sqrt{2}/\sqrt{3}}$.

Problem 4) (10 points)

a) (4 points) Write down the parameterization of the sphere $(x-1)^2 + (y-1)^2 + (z+2)^2 = 9$. using suitably centered spherical coordinates θ and ϕ.

b) (3 points) The latitudes on the sphere are curves defined by the equation $\phi = \textit{constant}$. Write down the parametric equations for the latitude $\phi = \pi/4$.

c) (3 points) Write down the arc length integral for this curve and evaluate it.

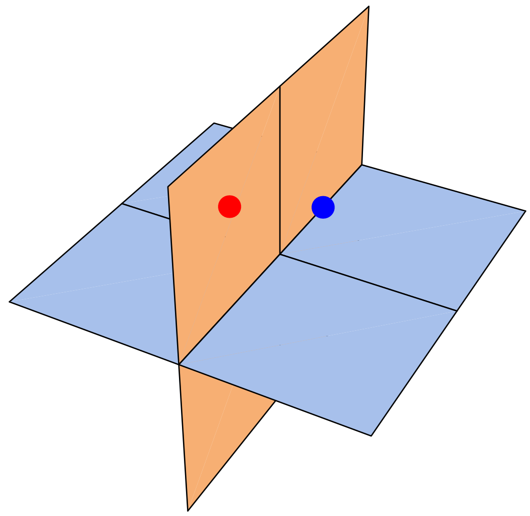
Solution:
a) $\vec{r}(\theta, \phi) = \langle 1 + 3\sin(\phi)\cos(\theta), 1 + 3\sin(\phi)\sin(\theta), -2 + 3\cos(\phi) \rangle$.
b) $\vec{r}(\theta) = \vec{r}(\theta, \pi/4) = \langle 1 + 3\cos(\theta)/\sqrt{2}, 1 + 3\sin(\theta)/\sqrt{2}, -2 + 3/\sqrt{2} \rangle$.
c) $\int_0^{2\pi} |\vec{r}(\theta)|\ d\theta = (3/\sqrt{2})2\pi = \boxed{3\sqrt{2}\pi}$.

Problem 5) (10 points)

Given the plane $x + y + z = 6$ containing the point $P = (2, 2, 2)$. Given is also a second point $Q = (3, -2, 2)$.

a) (5 points) Find the equation $ax + by + cz = d$ for the plane through P and Q which is perpendicular to the plane $x + y + z = 6$.

b) (5 points) Find the symmetric equation for the intersection of these two planes.



Solution:
a) The vector $\vec{v} = \langle 1,1,1\rangle$ is perpendicular to the first plane and so parallel to the second plane. The vector $\vec{w} = \vec{QP} = \langle 1,-4,0\rangle$ is also in the second plane. Therefore, $\vec{n} = \vec{v}\times\vec{w} = \langle 4,1,-5\rangle$ is perpendicular to the second plane. The plane has the equation $\boxed{4x + y - 5z = d = 0}$. The constant $d = 0$ was obtained here by plugging in a point like $P = (2,2,2)$.
b) To get the intersection line, construct the vector $\vec{n}\times\vec{v} = \langle 6,-9,3\rangle$ which is parallel to the line. The symmetric equation is
$\boxed{\frac{x-2}{6} = \frac{y-2}{-9} = \frac{z-2}{3}}$.

Problem 6) (10 points)

Intersecting the elliptic cylinder $x^2 + y^2/4 = 1$ with the plane $z = \sqrt{3}x$ gives a curve in space.

a) (3 points) Verify that this curve is parametrized by $\vec{r}(t) = \langle \sin(t), 2\cos(t), \sqrt{3}\sin(t)\rangle$ and give the parameter interval.

b) (3 points) Compute the unit tangent vector $\vec{T}$ to the curve at the point $(0,2,0)$.

c) (4 points) Write down the arc length integral and evaluate the arc length of the curve.

Solution:
a) With $x = \sin(t), y = 2\cos(t), z = \sqrt{3}\sin(t)$, we check $x^2 + y^2/4 = \sin^2(t) + \cos^2(t) = 1$.
b) Compute $\vec{r}\,'(t) = \cos(t), -2\sin(t), \sqrt{3}\cos(t)\rangle$ and
$\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,')(t)| = \langle\cos(t)/2, -\sin(t), \sqrt{3}\cos(t)\rangle$.
c) $|\vec{r}(t)| = 2$. The length is $\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} 2\,dt = \boxed{4\pi}$.

Problem 7) (10 points)

We know the acceleration $\vec{r}\,''(t) = \langle 2,1,3\rangle + t\langle 1,-1,1\rangle$ and the initial position $\vec{r}(0) = \langle 0,0,0\rangle$ and initial velocity $\vec{r}\,'(0) = \langle 0,0,0\rangle$ of an unknown curve $\vec{r}(t)$. Find $\vec{r}(100)$.

Solution:

$$\vec{r}\,'(t) = \int_0^t \langle 2+t, 1-t, 3+t\rangle\,dt + \vec{r}\,'(0) = \langle 2t + t^2/2, t - t^2/2, 3t + t^2/2\rangle$$

$$\vec{r}\,(t) = \int_0^t \langle 2t + t^2/2, t - t^2/2, 3t + t^2/2\rangle\,dt + \vec{r}(0) = \langle t^2 + t^3/6, t^2/2 - t^3/6, 3t^2/2 + t^3/6\rangle$$

Plug in the time $t = 100$ gives $= \boxed{10'000 \cdot \langle 106, -97, 109\rangle/6}$.

Problem 8) (10 points)

The elliptic paraboloid $f(x,y,z) = x^2 + 2y^2 - z = 0$ contains the point $(1,1,3)$.

a) (4 points) Find the equation for the tangent plane at $(1,1,3)$.

b) (3 points) Write down the linear approximation function $L(x,y,z)$ of $f(x,y,z)$ at $(1,1,3)$.

c) (3 points) Estimate $f(1.01, 1.0002, 2.999)$.

Solution:
a) Compute $f_x(x,y,z) = 2x, f_y(x,y,z) = 4y, f_z(x,y,z) = -1$. At the point $(1,1,3)$, we have $a = f_x(1,1,3) = 2, b = f_y(1,1,2) = 4, c = f_z(1,1,3) = -1$. The equation is $\boxed{2x + 4y - z = 3}$. The right hand side was obtained by pluggin in the point $(1,1,3)$.
b) $L(x,y,z) = f(1,1,3) + 2(x-1) + 4(y-1) - (z-3) = \boxed{2x + 4y - z - 3}$.
c) We estimate $f(1.01, 1.0002, 2.999)$ with $L(1.01, 1.0002, 2.999) = 2 \cdot 0.01 + 4 \cdot 0.0002 + 1 \cdot 0.001 = \boxed{0.0218}$.

Problem 9) (10 points)

a) (5 points) Show that the function $u(t,x) = \cos(2t - x) + \sin(2t - x)$ satisfies the **Klein-Gordon partial differential equation**

$$u_{tt} = u_{xx} - 3u \ .$$

b) (3 points) Describe the level curves of u.

c) (2 points) Parameterize one of the level curves $u(t,x) = 1$.

Solution:
a) Take the partial derivatives: $u_t(t,x) = -2\sin(2t - x) + 2\cos(2t - x)$, $u_x(t,x) = \sin(2t - x) - \cos(2t - x)$ and then the second partial derivatives. The term

$$u_{tt}(t,x) = -4\cos(2t - x) - 4\sin(2t - x)$$

is the same as the sum of

$$u_{xx}(t,x) = -\cos(2t - x) - \cos(2t - x)$$

$$-3u(t,x) = -3\cos(2t - x) - 3\cos(2t - x)$$

b) The level curves of u are the points where u is constant. These are the places where $2t - x$ is constant. $\boxed{\text{The level curves are lines.}}$
c) Chose the level curve to $u = 0$ which gies $2t - x = 0$. This line can be parametrized by $\boxed{\vec{r}(t) = (t, 2t)}$.