Name:

- Please mark the box to the left which lists your section.

- Do not detach pages from this exam packet or unstaple the packet.

| MWF 10 Samik Basu |
| MWF 10 Joachim Krieger |
| MWF 11 Matt Leingang |
| MWF 11 Veronique Godin |
| TTH 10 Oliver Knill |
| TTH 115 Thomas Lam |

- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.

- Please write neatly.

- Do not use notes, books, calculators, computers, or other electronic aids.

- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10

11A		10
12A		10
13A		10

11B		10
12B		10
13B		10

| Total: | | 140 |

Problem 1) True/False questions (20 points)

1) [**T**] [F]   For any two vectors $\vec{v}$ and $\vec{w}$ one has $\text{proj}_{\vec{v}}(\vec{v} \times \vec{w}) = \vec{0}$.

Solution:
The projection of a vector $\vec{v}$ onto a vector $\vec{w}$ which is perpendicular to $\vec{v}$ is zero.

2) [T] [**F**]   Any parameterized surface S is either the graph of a function $f(x, y)$ or a surface of revolution.

Solution:
A counter example is an asymetric ellipsoid. It is neither a graph, nor a surface of revolution.

3) [T] [**F**]   If the directional derivative $D_{\vec{v}}(f)$ of f into the direction of a unit vector $\vec{v}$ is zero, then $\vec{v}$ is perpendicular to the level curve of f.

Solution:
It is either tangent to the level curve or at a critical point.

4) [**T**] [F]   The linearization $L(x, y)$ of $f(x, y) = 5x - 100y$ at $(0, 0)$ satisfies $L(x, y) = 5x - 100y$.

Solution:
The linearization of any linear function at $(0, 0)$ is the function itself. This would be false if the point would not be $(0, 0)$.

5) [**T**] [F]   If a parameterized curve $\vec{r}(t)$ intersects a surface $\{f = c\}$ at a right angle, then at the point of intersection we have $\nabla f(\vec{r}(t)) \times \vec{r}'(t) = 0$.

Solution:
This is clear, once you know what the question means. The condition $\nabla f(\vec{r}(t)) \times \vec{r}'(t) = 0$ means that the velocity vector $\vec{r}'(t)$ is parallel to the gradient vector, which means that it is perpendicular to the level surface.

6) **T** F — The curvature of the curve $\vec{r}(t) = \langle \cos(3t^2), \sin(6t^2) \rangle$ at the point $\vec{r}(1)$ is larger than the curvature of the curve $\vec{r}(t) = \langle 2\cos(3t), 2\sin(6t) \rangle$ at the point $\vec{r}(1)$.

Solution:
While curvature is independent of the parameterization of the curve, the two circles have also a different radius. The second curve has twice the radius, so half the curvature.

7) **T** F — At every point (x, y, z) on the hyperboloid $x^2 - y^2 + z^2 = 10$, the vector $\langle x, -y, z \rangle$ is normal to the hyperboloid.

Solution:
Indeed, it is the gradient of the level surface. And the gradient is perpendicular to the level surface.

8) **T** F — The set $\{\phi = \pi/2, \theta = \pi/2\}$ in spherical coordinates is the positive y-axis.

Solution:
$\phi = \pi/2$ forces the vector to be on the xy-plane. The angle $\theta = \pi/2$ confines it to the y-axes.

9) T **F** — The integral $\int_0^1 \int_0^{2\pi} r^2 \sin(\theta) \; d\theta \, dr$ is equal to the area of the unit disk.

Solution:
We are using polar coordinates, not spherical coordinates in the plane. The correct integral is $\int_0^1 \int_0^{2\pi} r \; d\theta \, d$

10) **T** F — If three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ attached at the origin are in a common plane, then $\vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w}) = 0$.

Solution:
The volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$ is $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w})$. The volume is zero if and only if the parallelepiped is flat.

11) T **F** — If a function $f(x, y)$ has a local minimum at $(0, 0)$, then the discriminant D must be positive.

Solution:
False, we also can have $D = 0$ like for the function $f(x, y) = 1 - x^4 - y^4$.

12) T **F** — The integral $\int_0^1 \int_y^1 f(x, y) \, dy \, dx$ represents a double integral over a bounded region in the plane.

Solution:
The integral is not properly defined. There can be no variable in the most outer integral.

13) **T** F — The following identity is true: $\int_0^3 \int_0^x x^2 \, dy dx = \int_0^3 \int_y^3 x^2 \, dx \, dy$

Solution:
Make a picture and draw the triangle.

14) T **F** — There is a quadric, for which all three traces are hyperbola.

Solution:
Check through the list of all the quadrics. Paraboloids have at least one parabola as a trace. Hyperboloids have a circle or ellipse as a trace. Ellipsoids have ellipses as traces.

15) T **F** — The curvature of a space curve $\vec{r}(t)$ is a vector perpendicular to the acceleration vector $\vec{r}''(t)$.

Solution:
The curvature is a scalar, not a vector.

Solution:
While the point (x, y) is a critical point of $g(x, y) = x^2 + y^2 + f(x, y)$, it can be a local minimum. An example is $f(x, y) = 1 - (x^2 + y^2)/2$ so that the surface S is an ellipsoid $x^2/2 + y^2/2 + z^2 = 1$. Now $g(x, y) = 1 + (x^2 + y^2)/2$. The point on the surface with maximal distance to the origin is $(0, 0, 1)$. But $(0, 0)$ is a local minimum of $g(x, y)$. (This was the hardest TF problem in this exam.)

16) T **F**

Assume S is the unit sphere oriented so that the normal vector points outside. Let S^+ be the upper hemisphere and S^- the lower hemisphere. If a vector field $\vec{F}$ has divergence zero, then the flux of $\vec{F}$ through S^+ is equal to the flux of $\vec{F}$ through S^-.

Solution:
The sum of the fluxes is 0 so that flux through the total sphere is 0.

17) **T** F

If a vector field $\vec{F}$ is defined at all points in three-space except at the origin and $\text{curl}(\vec{F}) = \vec{0}$ everywhere, then the line integral of $\vec{F}$ around any closed path not passing through the origin is zero.

Solution:
The region without the origin is simply connected so that by Stokes theorem, one has a conservative vector field.

21) T **F**

If the sum of the variances of X and Y is equal to the variance of $X + Y$, then the random variables X and Y are independent.

Solution:
Only the converse of the statement is true. One can construct dependent random variables for which the sum of the variances is also the variance of $X + Y$. A simple example is given by constant random variables $X(x) = 0, Y(x) = 0$ for which the variance is zero for both and the sum of the variances is the variance of the sum.

18) **T** F

Every vector field which satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere in space can be written as $\vec{F} = \text{grad}(f)$ for some scalar function f.

Solution:
This is a consequence of Stokes theorem: for any closed curve find a surface which has this curve as a boundary. By Stokes theorem, the line integral is zero. The field consequently has the closed loop property and is so a gradient field.

22) **T** F

Let X be a random variable with zero mean $E(X) = 0$. Then the expectation $E(X^2)$ is equal to the variance $D(X)$.

Solution:
We have $D(X) = E(X^2) - E(X)^2$.

19) **T** F

Let $\vec{F}$ be a vector field and let S be an oriented surface $\vec{r}(u, v)$. Then $20 \int \int_S \vec{F} \cdot dS = \int \int_S \vec{G} \cdot dS$, where $\vec{G} = 20\vec{F}$.

Solution:
The integral is linear.

23) **T** F

If A, B are two events which have positive probability and $P(A|B)$ as well as $P(B|A)$ are known, then we can compute $P(A)/P(B)$.

Solution:
Use the definitions $P(A|B) = P(A \cap B)/P(B)$ and $P(B|A) = P(A \cap B)/P(A)$.

20) T **F**

Consider the surface S given by the equation $z^2 = f(x, y)$. If $(x, y, z) = (x, y, \sqrt{f(x, y)})$ is a point on the surface with maximal distance from the origin, it is a local maximum of $g(x, y) = x^2 + y^2 + f(x, y)$.

24) T **F** The function $f(x) = e^{-x}$ on $[0, \infty)$ is the distribution function of a random variable.

Solution:
It is not a distribution function, it is a density function.

25) T **F** Suppose you throw two fair coins n times. The probability to have k times head is $k!(n-k)!/2^n$.
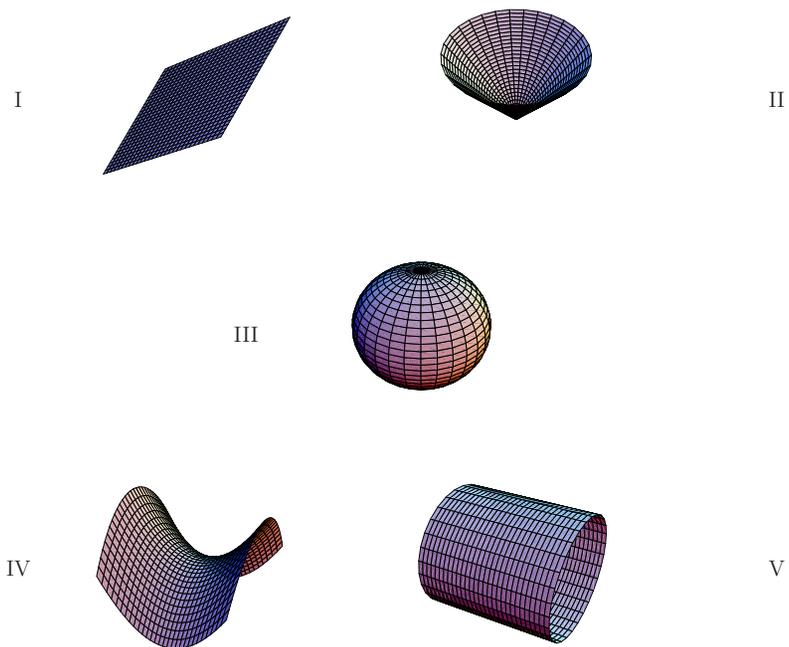
Solution:
There is a $n!$ in the denominator missing.

Problem 2) (10 points)

Match the parameterized surface formulas and pictures with the formulas for the implicit surfaces. No justifications are needed.

A)	$\vec{r}(u,v) = \langle 1 + u, v, u + v \rangle$
B)	$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v \rangle$
C)	$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$
D)	$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$
E)	$\vec{r}(u,v) = \langle v, \sin(u), \cos(u) \rangle$

I

II

III

IV

V

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
		$y^2 + z^2 = 1$
		$x + y - z = 1$
		$x^2 + y^2 + z^2 = 1$
		$x^2 + y^2 - z^2 = 0$
		$x^2 - y^2 - z = 0$

Solution:

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
E	V	$y^2 + z^2 = 1$
A	I	$x + y - z = 1$
C	III	$x^2 + y^2 + z^2 = 1$
B	II	$x^2 + y^2 - z^2 = 0$
D	IV	$x^2 - y^2 - z = 0$

Problem 3) (10 points)

Match the formulas and theorems with their names. No justifications are needed.

N) $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$
U) $f_{xy}(x,y) = f_{yx}(y,x)$
M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$
E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$
L) $(\vec{u}\times\vec{v})\cdot\vec{w}$
F) $|\vec{v}\cdot\vec{w}| \le |\vec{v}|\cdot|\vec{w}|$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$
G) $|\vec{v}+\vec{w}| \le |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
	Fubini theorem
	Clairot theorem
	vector projection
	scalar projection
	chain rule
	dot product formula
	scalar triple product
	Cauchy-Schwarz inequality
	Pythagorean theorem
	Triangle inequality

Solution:
The solution is unscambled. [Originally, we planned to make a nice anagram, but somehow, the scrambling was only done in the solutions. Here had been the cool anagram NUMERAL FIG → FUELING ARM:]
F) $|\vec{v}\cdot\vec{w}| \le |\vec{v}|\cdot|\vec{w}|$.
U) $f_{xy}(x,y) = f_{yx}(x,y)$.
E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$
L) $(\vec{u}\times\vec{v})\cdot\vec{w}$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$.
N) $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$.
G) $|\vec{v}+\vec{w}| \le |\vec{v}| + |\vec{w}|$.
A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}w$.

The actual solution to the problem posed in the exam was that each letter stayed at the height it had been before:

N) $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$
U) $f_{xy}(x,y) = f_{yx}(y,x)$
M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$
E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$
L) $(\vec{u}\times\vec{v})\cdot\vec{w}$
F) $|\vec{v}\cdot\vec{w}| \le |\vec{v}|\cdot|\vec{w}|$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$
G) $|\vec{v}+\vec{w}| \le |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
N	Fubini theorem
U	Clairot theorem
M	vector projection
E	scalar projection
R	chain rule
A	dot product formula
L	scalar triple product
F	Cauchy-Schwarz inequality
I	Pythagorean theorem
G	Triangle inequality

Problem 4) (10 points)

Let L be the line $\vec{r}(t) = \langle t, 0, 0\rangle$. We are also given a point $Q = (3,3,0)$ in space.

a) (2 points) What is the distance $d((x,y,z), Q)$ between a general point (x,y,z) and Q?

b) (3 points) What is the distance $d((x,y,z), L)$ between the point (x,y,z) and the line L?

c) (3 points) Find the equation for the set C of all points (x,y,z) satisfying

$$d((x,y,z), Q) = d((x,y,z), L).$$

d) (2 points) Identify the surface.

Solution:

a) $\sqrt{(x-3)^2 + (y-3)^2 + z^2}$.

b) The distance formula is $|\langle 1, 0, 0\rangle \times \langle (x, y, z)\rangle|/\sqrt{1} = |\langle -y, x - z, y\rangle| = \sqrt{y^2 + z^2}$.

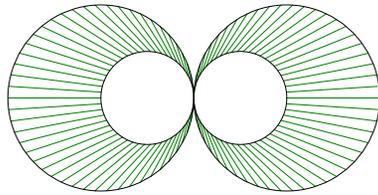c) The equation $y^2 + z^2 = (x-3)^2 + (y-3)^2 + z^2$ can be simplified to $\boxed{(x-3)^2 = 6y - 9}$.

d) This is a $\boxed{\text{cylindrical paraboloid}}$. It has the same shape as $x^2 = y$ but is translated and scaled in the y direction.

Problem 5) (10 points)

Find the area of the region in the plane given in polar coordinates by

$$\{(r, \theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi \}.$$

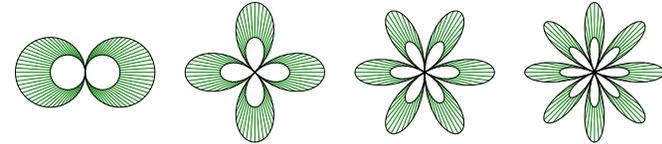The region is the shaded part in the figure.

Solution:

$\int_0^{2\pi} \int_{|\cos(\theta)|}^{|2\cos(\theta)|} r \, dr d\theta = \int_0^{2\pi} 4\cos^2(\theta)/2 - \cos^2(\theta)/2 \, d\theta = \boxed{3\pi/2}$. The result is the same for any region

$$\{(r, \theta) \mid |\cos(n\theta)| \leq r \leq 2|\cos(n\theta)|, 0 \leq \theta < 2\pi \}.$$

(The graphics had originally been shown in case $n = 2$, not $n = 1$. If this had been a source for an error, we would not penalize it in our grading.)
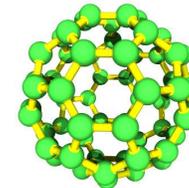

n=1   n=2   n=3   n=4

Problem 6) (10 points)

A microscopic bucky ball C60 is located on a gold surface. The surface produces the electric potential $f(x, y) = x^4 + y^4 - 2x^2 - 8y^2 + 5$.

a) (7 points) Find all critical points of f and classify them.

b) (3 points) The fullerene will settle at a global minimum of $f(x, y)$. Find the global minima of the function $f(x, y)$.
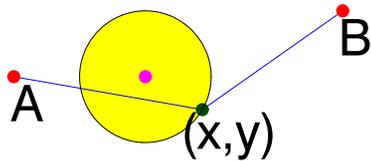
Solution:

a) The gradient of $f(x, y)$ is $\nabla f(x, y) = \langle x^3 - 4x, 4y^3 - 16y \rangle$. It is zero if $x = 0, 1, -1$ and $y = 0, 2, -2$. There are 9 critical points. The discriminant is $D = 2(3x^2 - 1)(3y^2 - 4)$ and and $X = f_{xx} = 12x^2 - 4 = 4(3x^2 - 1)$. Note that f_{xx} is negative if $x = 0$ and positive in all other cases.

point	D	f_{xx}	nature	value
(-1, -2)	256	8	min	-12
(-1, 0)	-128	8	saddle	4
(-1, 2)	256	8	min	-12
(0, -2)	-128	-4	saddle	-11
(0, 0)	64	-4	max	5
(0, 2)	-128	-4	saddle	-11
(1, -2)	256	8	min	-12
(1, 0)	-128	8	saddle	4
(1, 2)	256	8	min	-12

b) The points of global minima are $(1, 2), (1, -2), (-1, 2), (-1, -2)$. The global minimum value is $\boxed{-12}$.

A circular wheel with boundary $g(x, y) = x^2 + y^2 = 1$ has the boundary point (x, y) connected to two points $A = (-2, 0)$ and $B = (3, 1)$ by rubber bands. The potential energy at position (x, y) is by Hooks law equal to $f(x, y) = (x+2)^2 + y^2 + (x-3)^2 + (y-1)^2$, the sum of the squares of the distances to A and B. Our goal is to find the position (x, y) for which the energy is minimal. To find this position for which the wheel is at rest, minimize $f(x, y)$ under the constraint $g(x, y) = 1$.

Solution:

The Lagrange equations

$$2(x + 2) + 2(x - 3) = \lambda 2x$$
$$2y + 2(y - 1) = \lambda 2y$$
$$x^2 + y^2 = 1$$

simplify to

$$2x - 1 = \lambda x$$
$$2y - 1 = \lambda y$$
$$x^2 + y^2 = 1$$

Dividing the first equation by the second gives $x = y$. Plugging this into the third equation gives $x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{2}$. For $(x, y) = (\sqrt{2}, \sqrt{2})$ we have $f(x, y) = 16 - 2\sqrt{2}$ and for $(x, y) = (-\sqrt{2}, -\sqrt{2})$ we have $f(x, y) = 16 + 2\sqrt{2}$. The minimum is at $\boxed{(\sqrt{2}, \sqrt{2})/2}$.

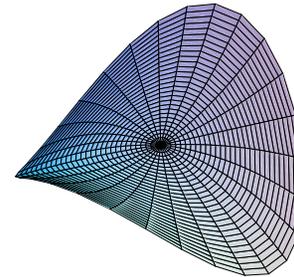a) (5 points) Find the surface area of the parameterized surface

$$\vec{r}(u, v) = \langle u - v, u + v, uv \rangle$$

with $u^2 + v^2 \le 1$.

b) (3 points) Find an implicit equation $g(x, y, z) = 0$ for this surface.
Hint: Look at $y^2 - x^2$.

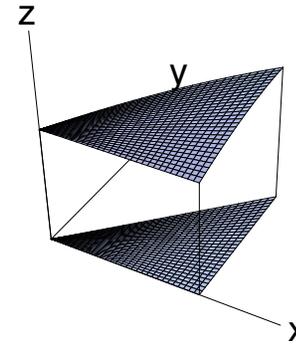c) (2 points) What is the name of the surface?

Solution:
a) $\vec{r}_u \times \vec{r}_v = \langle u - v, -u - v, 2 \rangle$ and $|\vec{r}_u \times \vec{r}_v| = \sqrt{4 + 2(u^2 + v^2)}$. So, $\int \int |\vec{r}_u \times \vec{r}_v| \, dudv =$
$\int_0^1 \int_0^{2\pi} \sqrt{2}\sqrt{2 + r^2} r \, d\theta dr = \sqrt{2} 2\pi(\sqrt{3} - 2\sqrt{2}/3)$.
The answer is $\boxed{\pi 3/(6^{3/2} - 4^{3/2})}$.
b) $x^2 - y^2 = 4z$.
c) This is a hyperbolic paraboloid, also known as "saddle" or "pringle".

Problem 9) (10 points)

a) (4 points) Find the tangent plane to the surface $f(x, y, z) = zx^5 + y^5 - z^5 = 1$ at the point $(1, 1, 1)$.

b) (3 points) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at at the point $(1, 1, 1)$.

c) (3 points) Near the point $(1, 1, 1)$, the surface can be written as a graph $z = g(x, y)$. Find the partial derivative $g_x(1, 1)$.

Solution:
a) The gradient at a general point is $\nabla f(x, y, z) = \langle 5x^4 z, 5y^4, x^5 - 5z^4 \rangle$.
So, the gradient at the point $(1, 1, 1)$ is equal to $\nabla f(1, 1, 1) = \langle 5, 5, -4 \rangle = \langle a, b, c \rangle$. The equation of the plane is $ax + by + cz = d$ where the constant d is obtained by plugging in plugging in one point. Here $\boxed{5x + 5y - 4z = 6}$.
b) The linearization is defined as $L(x, y, z) = f(1, 1, 1) + f_x(1, 1, 1)(x - 1) + f_y(1, 1, 1)(y - 1) + f_z(1, 1, 1)(z - 1) = 1 + 5(x - 1) + 5(y - 1) - 4(z - 1)$. This can be simplified to $\boxed{L(x, y, z) = 5x + 5y - 4z - 5}$.
c) The implicit differentiation formula derived from the chain rule gives $g_x(1, 1) = -f_x(1, 1, 1)/f_z(1, 1, 1) = \boxed{5/4}$. (This can also be seen as the slope of the xz-trace of the tangent plane.)

Problem 10) (10 points)

A tower E with base $0 \le x \le 1, 0 \le y \le x$ has a roof $f(x, y) = \sin(1 - y)/(1 - y)$. Find the volume of this solid. The solid is given in formulas by

$$E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le f(x, y)\}.$$

Solution:
The type I integral
$$\int_0^1 \int_0^x \frac{\sin(1 - y)}{(1 - y)} \, dydx$$
can not be solved. We have to change the order of integration:
$$\int_0^1 \int_y^1 \frac{\sin(1 - y)}{(1 - y)} \, dxdy = \int_0^1 \sin(1 - y) \, dy = \cos(1 - y)\big|_0^1 = 1 - \cos(1) .$$

The final result is $\boxed{1 - \cos(1)}$.

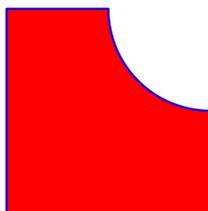**The following problem 11 A is for regular sections only:**

Problem 11 A) (10 points)

Let $\vec{F} = \langle y, 2x + \tan(\tan(y)) \rangle$ be a vector field in the plane and let C be the boundary of the region
$$G = \{0 \le x \le 2, 0 \le y \le 2, (x - 2)^2 + (y - 2)^2 \ge 1\}$$
oriented counter clock-wise. Compute the line integral
$$\int_C \vec{F} \cdot d\vec{r} .$$

Solution:
The curl of $\vec{F} = \langle P, Q \rangle$ is constant $Q_x - P_y = 2 - 1 = 1$. The line integral by **Greens theorem**

$$\int_C \vec{F} \, d\vec{r} = \int\int_G \vec{F} \, dA$$

equal to the area of the region, which is the area of the square minus the area of the quarter disc: $\boxed{4 - \pi/4}$.

The following problem 12 A is for regular sections only:

Problem 12 A) (10 points)

Let S be the surface of a turbine blade parameterized by $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), t \rangle$ for $t \in [0, 6\pi]$ and $s \in [0, 1]$. Let $\vec{F} = \mathrm{curl}(\vec{G})$ denote the velocity field of the water velocity, where $\vec{G}(x, y, z) = \langle -y + (x^2 + y^2 - 1), x + (x^2 + y^2 - 1), (x^2 + y^2 - 1) \rangle$. Compute the power of the turbine which is given by the flux of $\vec{F} = \mathrm{curl}(\vec{G})$ through S.
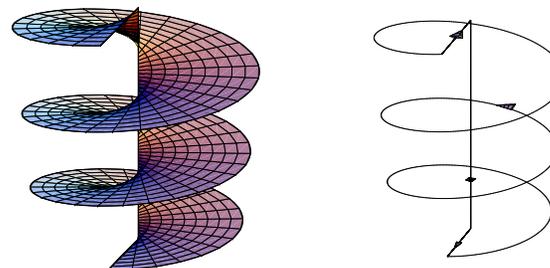
Hint. The boundary C of the surface S consists of 4 paths:
$\vec{r_1}(t) = \langle \cos(t), \sin(t), t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_2}(s) = \langle 1 - s, 0, 6\pi \rangle$, $s \in [0, 1]$.
$\vec{r_3}(t) = \langle 0, 0, 6\pi - t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_4}(s) = \langle s, 0, 0 \rangle$, $s \in [0, 1]$.

Solution:
We use **Stokes theorem**

$$\int\int_S \mathrm{curl}(\vec{G}) \, dS = \int_C \vec{G} \, d\vec{r} \ .$$

This allows us to compute the line integral along the boundary instead of the flux. This line integral consists of 4 paths. The line integrals along $\vec{r_2}$ and $\vec{r_4}$ cancel, one is $2/3$, the other $-2/3$. On $\vec{r_1}$, the vector field is $\vec{G}(x, y, z) = \langle -y, x, 0 \rangle$ and $\vec{G}(\vec{r}(t)) = \langle -\sin(t), \cos(t), 0 \rangle$ is parallel to $\vec{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. Computing the line integral gives $\int_0^{6\pi} 1 \, dt = 6\pi$. On $\vec{r_3}$, both x and y are zero so that the vector field is $\langle -1, -1, -1 \rangle$ and the line integral gives $\int_0^{6\pi} \langle -1, -1, -1 \rangle \cdot \langle 0, 0, -1 \rangle \, dt = 6\pi$. The sum of all four line integrals is $\boxed{12\pi}$.

The following problem 13 A is for regular sections only:

Problem 13 A) (10 points)

Let $\vec{F}(x, y, z) = \langle z^2, -z^5 + z\sin(e^{\sin(x)}), (x^2 + y^2) \rangle$. Let S denote the part of the graph $z = 9 - x^2 - y^2$ lying above the xy-plane oriented so that the normal vector points upwards. Find the flux of $\vec{F}$ through the surface S.

Hint. You might also want to look at the surface $D = \{x^2 + y^2 \le 9, z = 0 \}$ lying in the xy-plane.

Solution:

The vector field is incompressible. By the **divergence theorem**, if D is oriented upwards, we have $\int \int_S \vec{F} \, dS - \int \int_D \vec{F} \, dS = 0$. The flux through S is therefore the same as the flux through D if both D and S are oriented upwards. The flux through D can be computed easily because $\vec{F} = \langle 0, 0, x^2 + y^2 \rangle$ there. That integral is best evaluated in polar coordinates $\int_0^3 \int_0^{2\pi} r^3 \, d\theta dr = 3^4 2\pi/4 = \boxed{81\pi/2}$.

The following problem 11 B is for biochem sections only:

Problem 11 B) (10 points)

A certain state has license plates with 6 entries. These entries can be letters (26 of them) and digits (10 or them). How many license plates are there under the following restrictions?

a) (3 points) Assume that we allow any combination of letters and digits.

b) (3 points) Assume that the license plates start with three letters and then have three digits.

c) (4 points) Assume, as in a), that we allow any combination of letters and digits. Assume that all plates are as likely. What are the odds of getting four or more letters on your plate?

Solution:

a) There are 36 possibilities for each entry : 10 digits and 26 letters. There are therefore

$$36 \times 36 \times 36 \times 36 \times 36 \times 36 = 36^6$$

different plates. Note that by shuffling the entries on the plate, we get a different plate and also that we are allowed to use the same entry as often as we pleased. b) There are 26 choices for each of the first three entries and 10 for each of the following three. Hence

$$26^3 \times 10^3 = 260^3$$

different such plates. c) The odds of getting a letter is

$$p = \frac{26}{36}.$$

Using the binomial formula, we get that the odds of getting at least four letters are

$$\sum_{k=4}^{6} \binom{6}{k} \left(\frac{26}{36}\right)^k \left(\frac{10}{36}\right)^{6-k}.$$

The following problem 12 B is for biochem sections only:

Problem 12 B) (10 points)

We have 10 coins. Eight of them are fair. Two coins are not fair: one coin has two tails and one has two heads. Say I pick a coin randomly from these ten coins and that I throw this coin six times.

a) (4 points) What is the probability that the chosen coin is the one with two tails if I obtained six tails?

b) (4 points) Are the following events independent? My first throw turns out to be a tail. My second throw is a head.

c) (2 points) Explain to a non-mathematician what your answer in b) means and why you would expect it.

Solution:

a) Let B_1 be the event 'I have picked a tail-only coin'. Let B_2 be 'I have picked an honest coin' and B_3 be 'I have picked a head-only'. Let A be the event 'I get six tails'. By Bayes' formula

$$
\begin{aligned}
P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{\sum_k P(A|B_k)P(B_k)} \\
&= \frac{1 \cdot 2/10}{1 \cdot 2/10 + (1/2)^6 \cdot 7/10 + 0 \cdot 1/10} \\
&= \frac{2}{2 + 7/(64)} = \frac{128}{128 + 7} = \frac{128}{135}
\end{aligned}
$$

. The final answer is $\boxed{128/135}$.

b) Let the events B_i be as before. Let A be my first throw is a tail and C be my second throw is a head. Again using Bayes' rule, we get

$$
\begin{aligned}
P(A) &= \sum_k P(A|B_k)P(B_k) = 1 \cdot 2/10 + 1/2 \cdot 7/10 + 0 \cdot 1/10 = 11/20 \\
P(C) &= 0 \cdot 2/10 + 1/2 \cdot 7/10 + 1 \cdot 1/10 = 9/20 \\
P(A \cap C) &= 0 \cdot 2/10 + 1/4 \cdot 7/10 + 0 \cdot 1/10 = 7/40 .
\end{aligned}
$$

Hence

$$P(A \cap C) \neq P(A)P(C)$$

which means that A and C are $\boxed{\text{dependent}}$.

c) Once we know that the first throw gives a head, we have eliminated the tail-only coin and made the head-only more likely. Hence if the first throw gives a head, the second throw will more likely give a head and C is less likely. This means that the outcome of the event A gives information about the outcome of the event C (and vice versa).

The following problem 13 B is for biochem sections only:

Problem 13 B) (10 points)

Taxis and buses travel from Central Square to Harvard square. Their travel time in minutes are independent random variables T and B with probability densities

$$p_T(t) = \begin{cases} 2e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad p_B(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

a) (3 points) Find the expected travel time to Harvard square both in a taxi and in a bus.

b) (3 points) What is the probability that a bus trip to Harvard square takes between 5 and 10 minutes?

c) (4 points) Peter is at Harvard square. Andrew and Kevin have a message to give to him. They leave Central square at the same time, but Andrew takes a bus and Kevin a taxi. Let Y be the number of minutes before Peter gets the message. Hence Y is the time before the **first** one of Andrew and Kevin arrives in Harvard square. What is the probability density of the random variable Y?

Solution:

a) Using integration by parts, we get

$$E(T) = \int_0^\infty 3te^{-3t}dt = \frac{1}{3}$$

$$E(B) = \int_0^\infty te^{-t}dt = 1.$$

b) $P\{5 \leq B \leq 10\} = \int_5^{10} e^{-t}dt = e^{-5} - e^{-10}$.

$Y = \min(T, B)$. Y is equal to t_0 in either of the following two cases

$$\{T = t_0 \,\&\, B = x \geq t_0\} \qquad \{B = t_0 \,\&\, T = x \geq t_0\}.$$

Hence

$$p_Y(t_0) = \int_{t_0}^\infty p_T(t_0)p_B(x)dx + \int_{t_0}^\infty p_B(t_0)p_T(x)\,dx$$

$$= \int_{t_0}^\infty \left(3e^{-3t_0}e^{-x} + e^{-t_0}3e^{-3x}\right)dx = 4e^{-4t}.$$