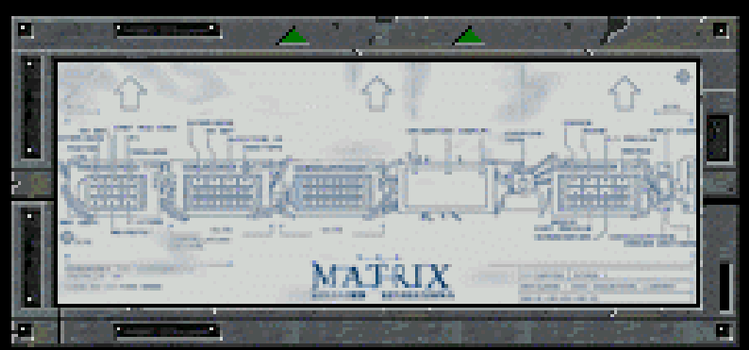
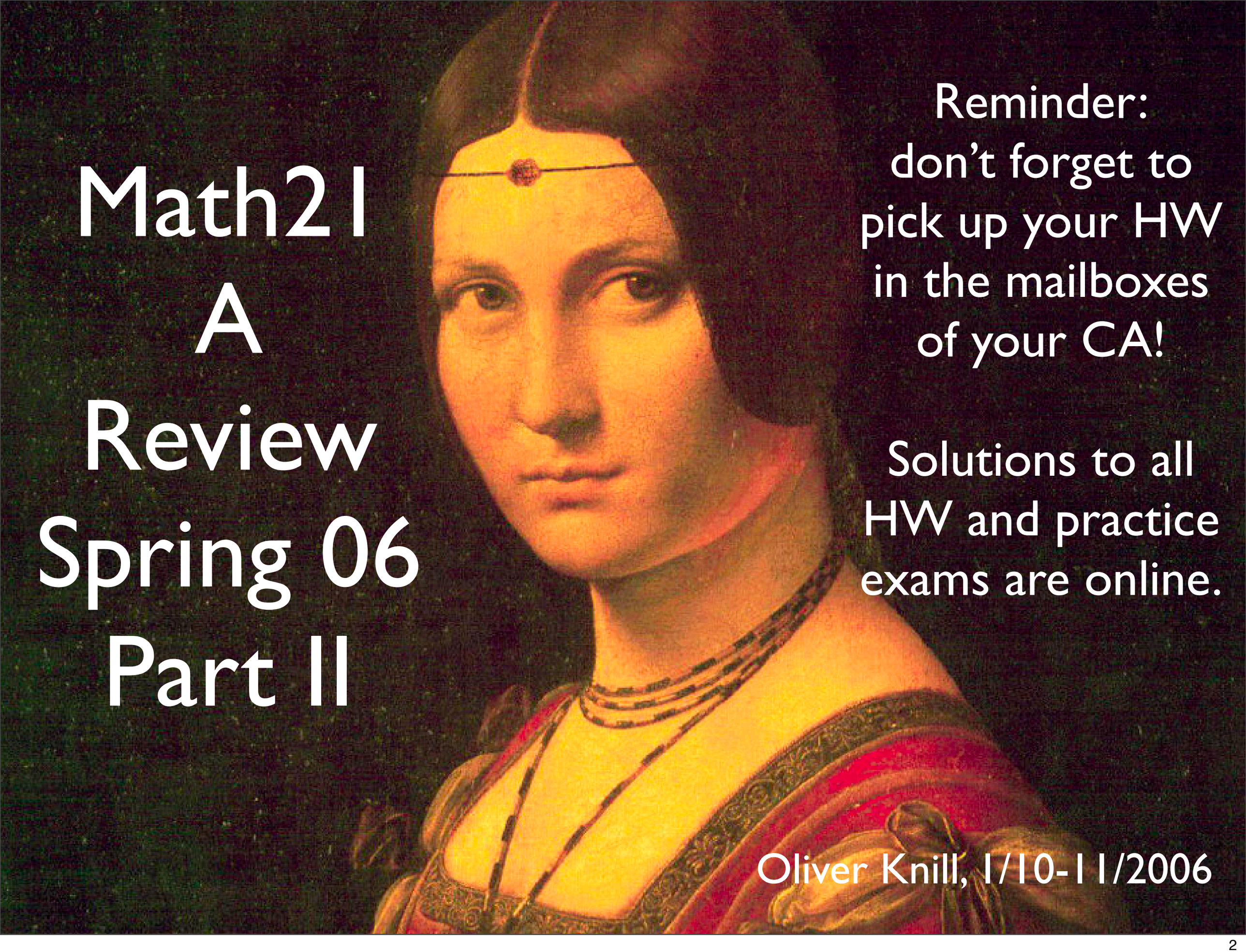
Math21a



Review

Oliver Knill, 2006

Math21 A Review Spring 06 Part II



Reminder: don't forget to pick up your HW in the mailboxes of your CA!

Solutions to all HW and practice exams are online.

Oliver Knill, 1/10-11/2006

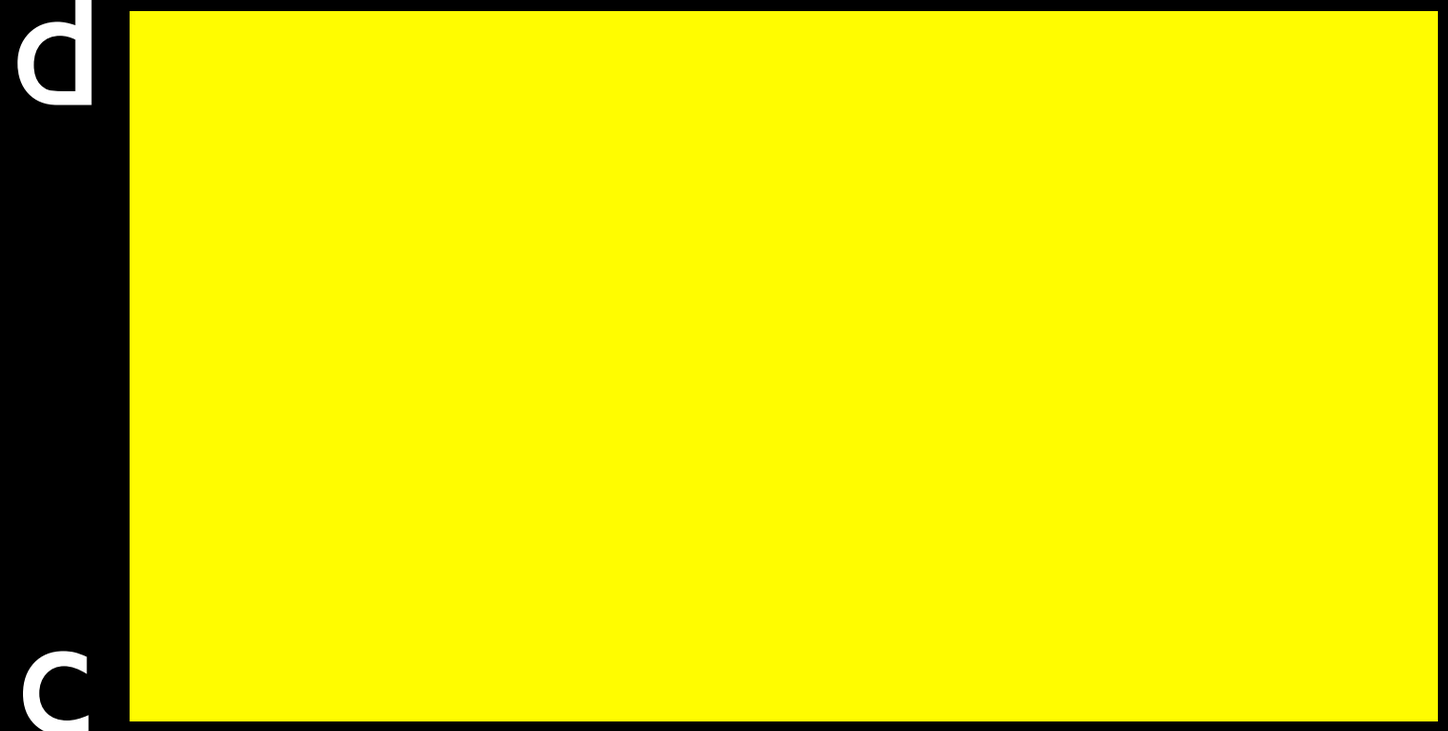
Part II: integration, integral theorems

From Leonardo to Harvard

We first cover integration

Start with Double Integrals

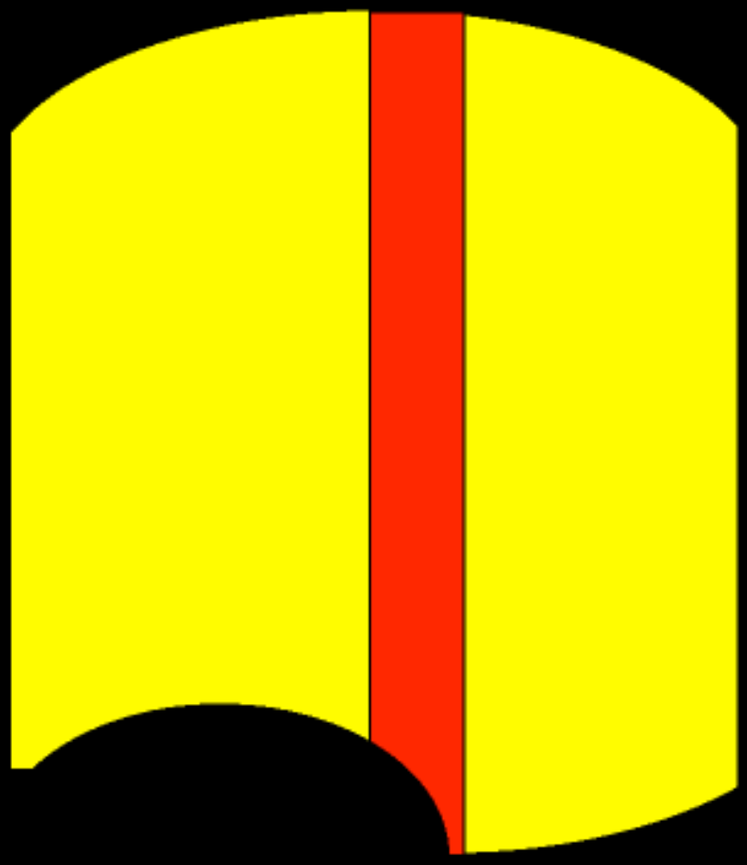
Fubini



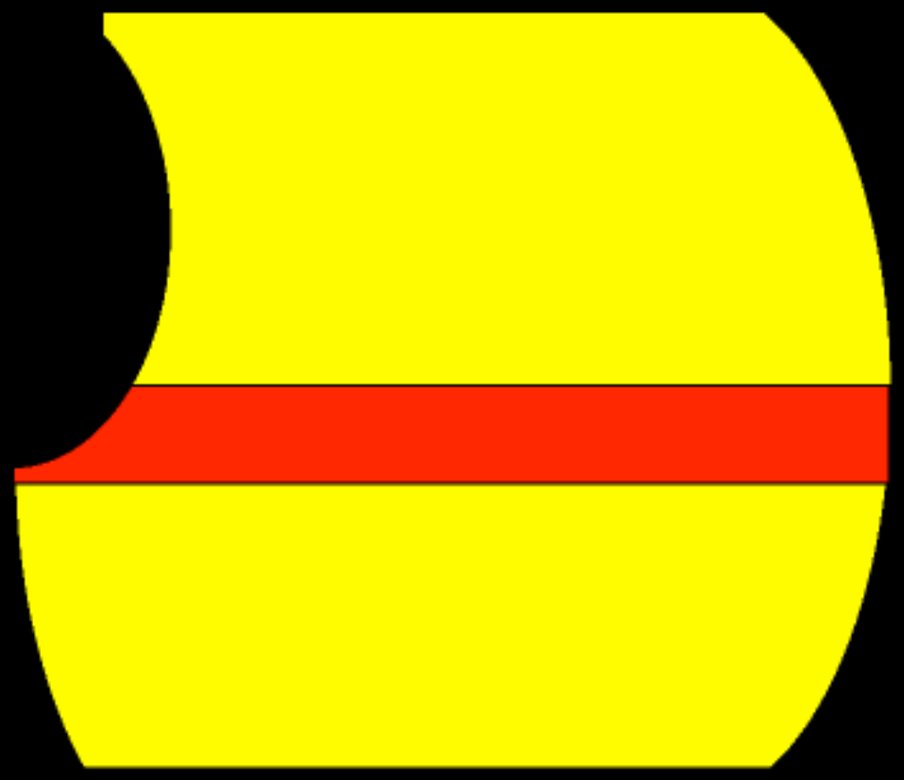
$$\int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$$

Double Integrals

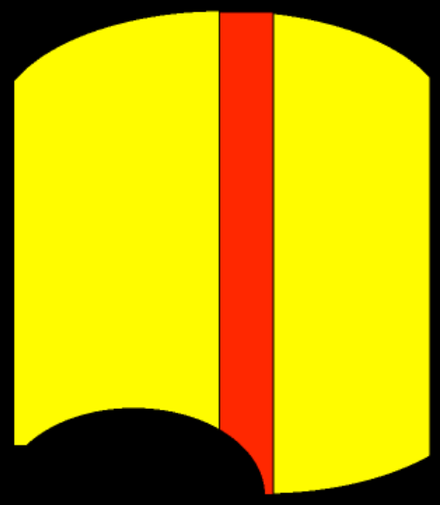
Type I

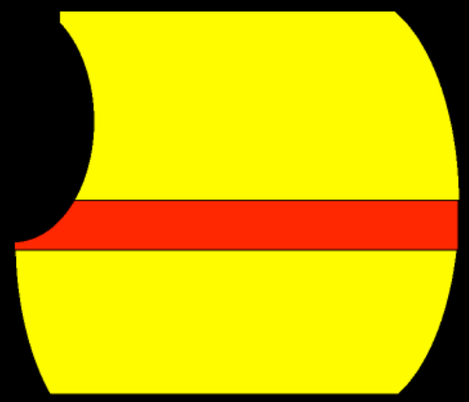


Type II



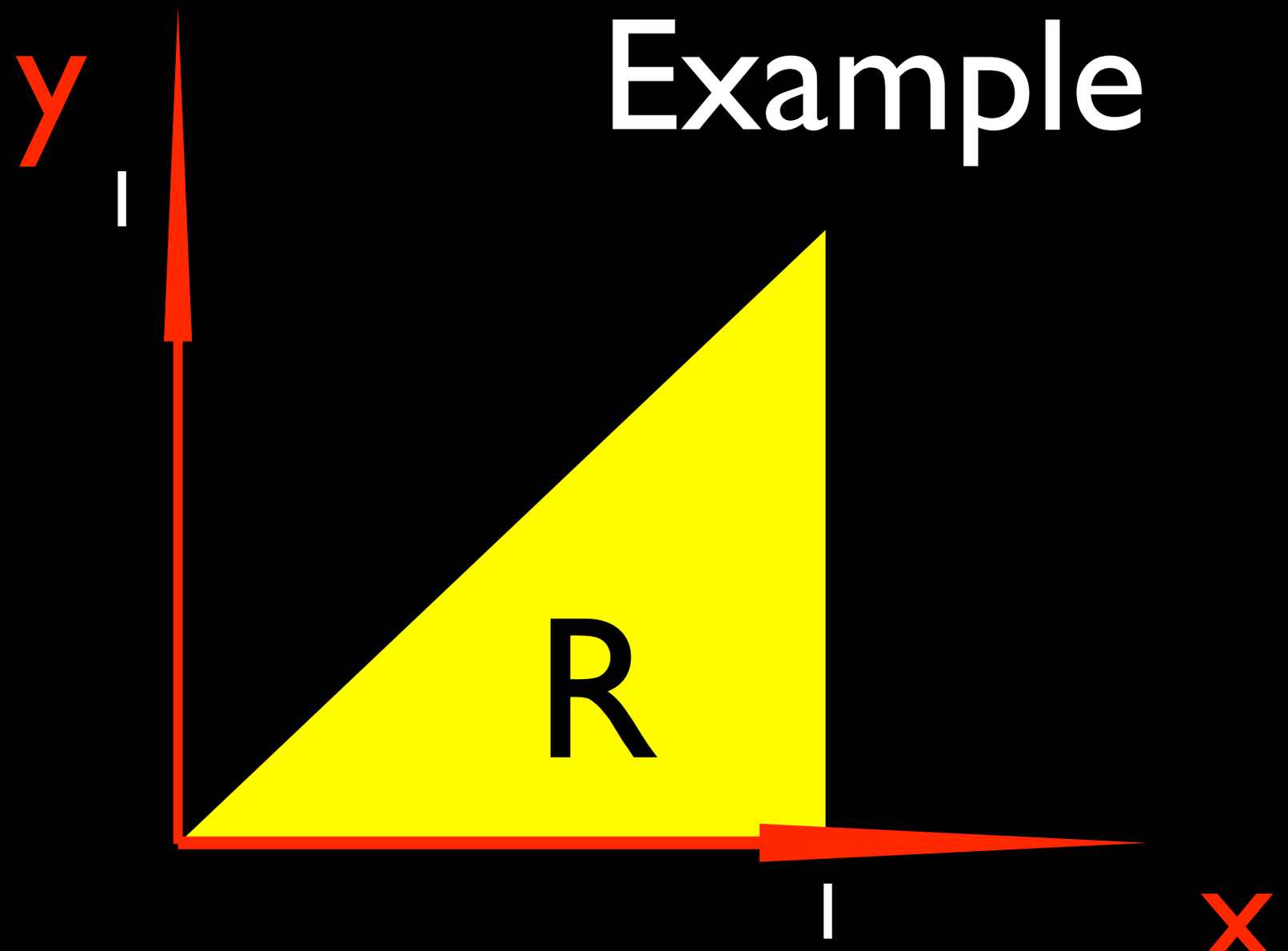
Change Type I to Type II



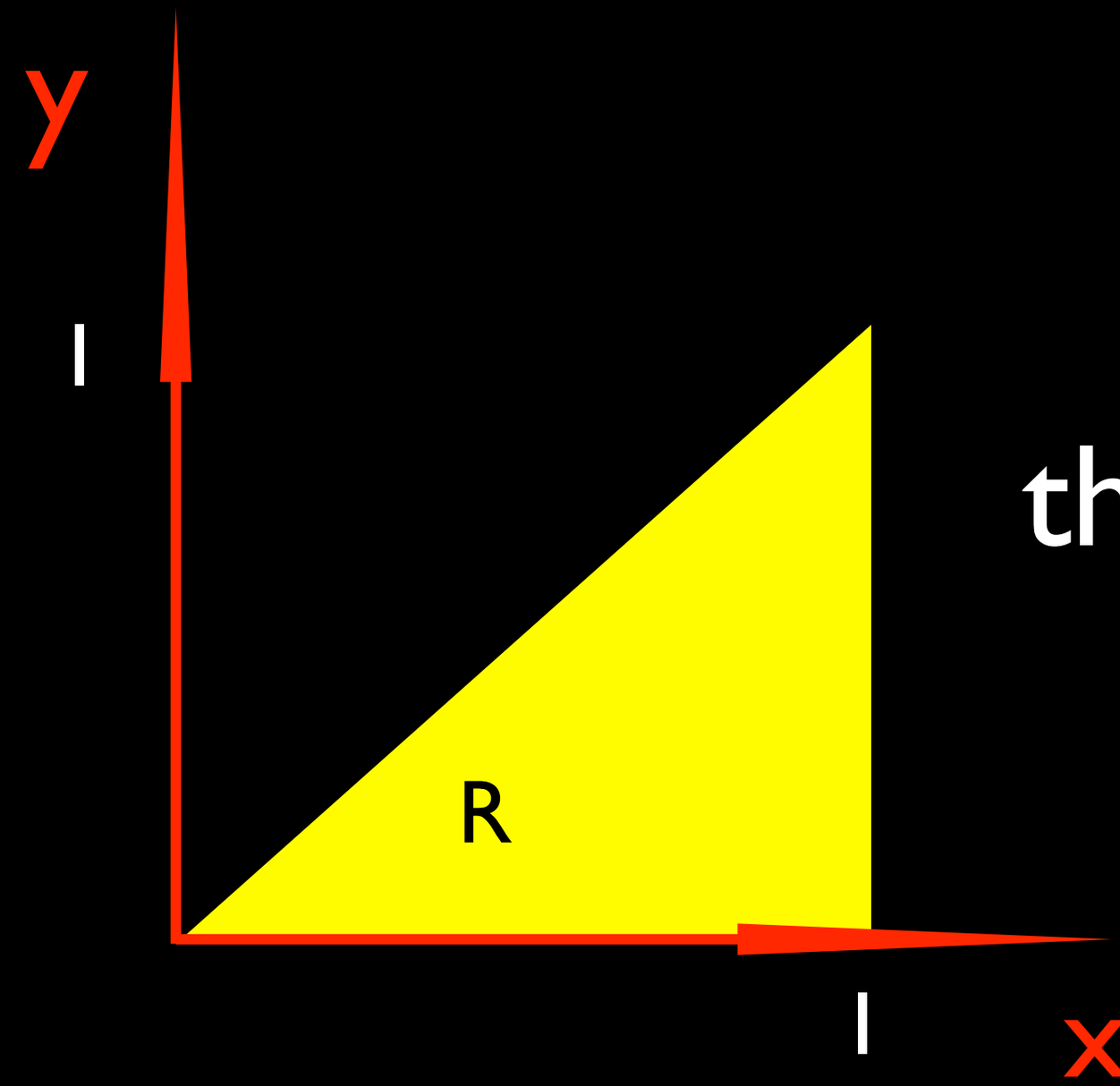


If you can not solve an integral as a type I integral, try to treat it as as type II integral

Example



Find: $\int_0^1 [\int_y^1 e^{-x^2}\, dx]\, dy$

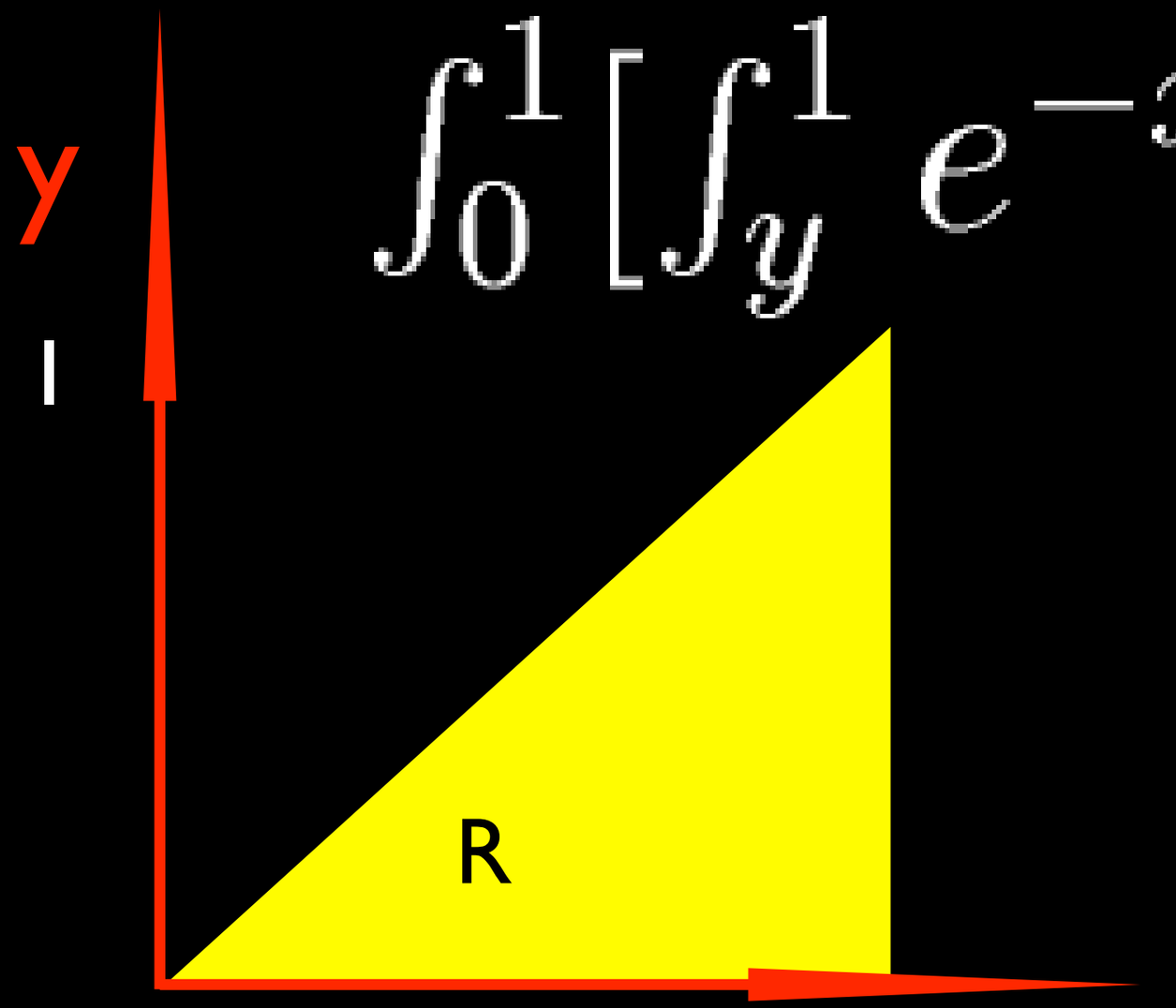


this is a Type II integral

$$\int_0^1 \left[\int_y^1 e^{-x^2}\, dx\right] dy$$

Are stuck!

$$\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$$



Change order of integration:

$$\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx$$

$$\int_0^1 x e^{-x^2}\ dx = -\frac{e^{-x^2}}{2}\Big|_0^1 = \frac{(1-e^{-1})}{2}$$

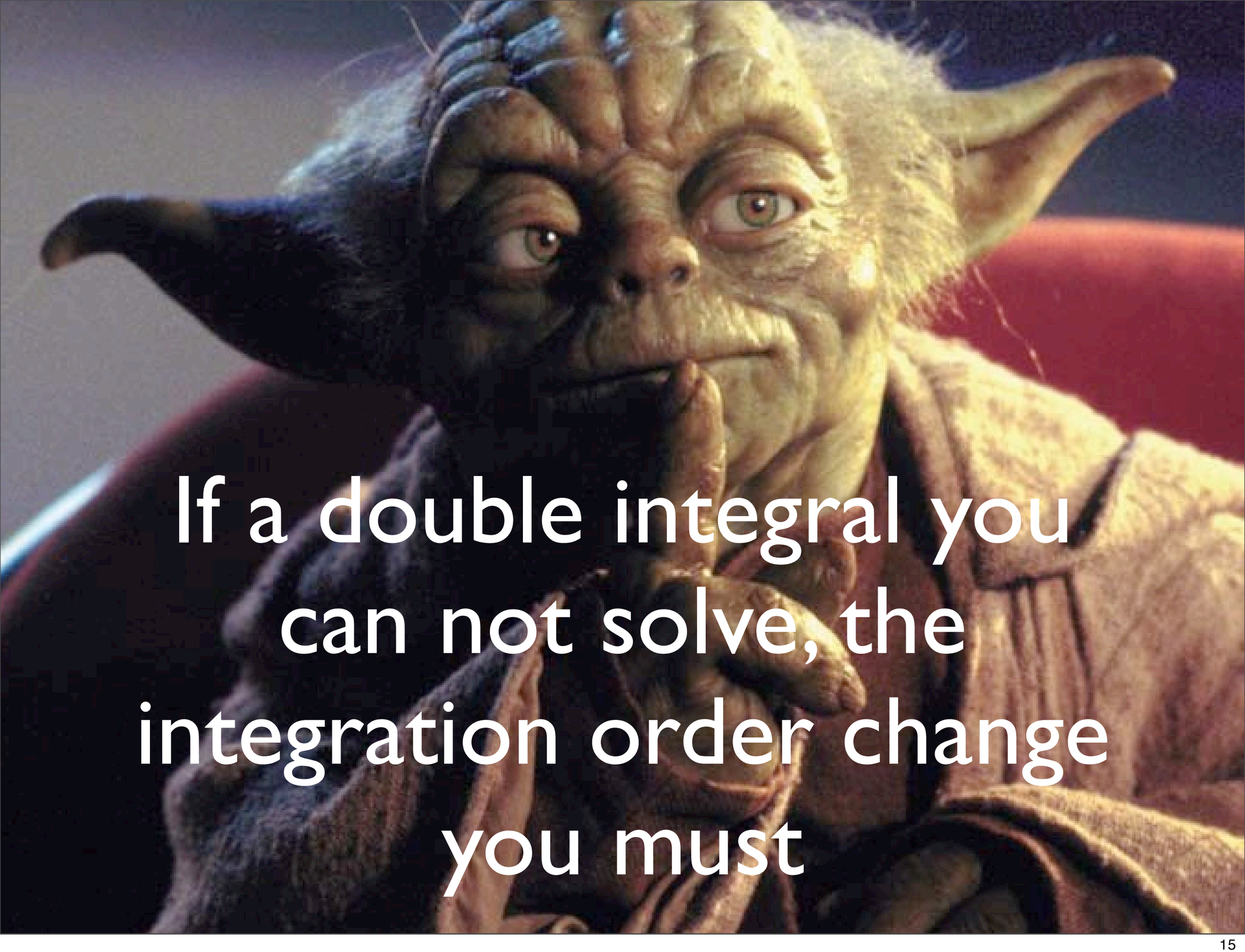
Example: What is

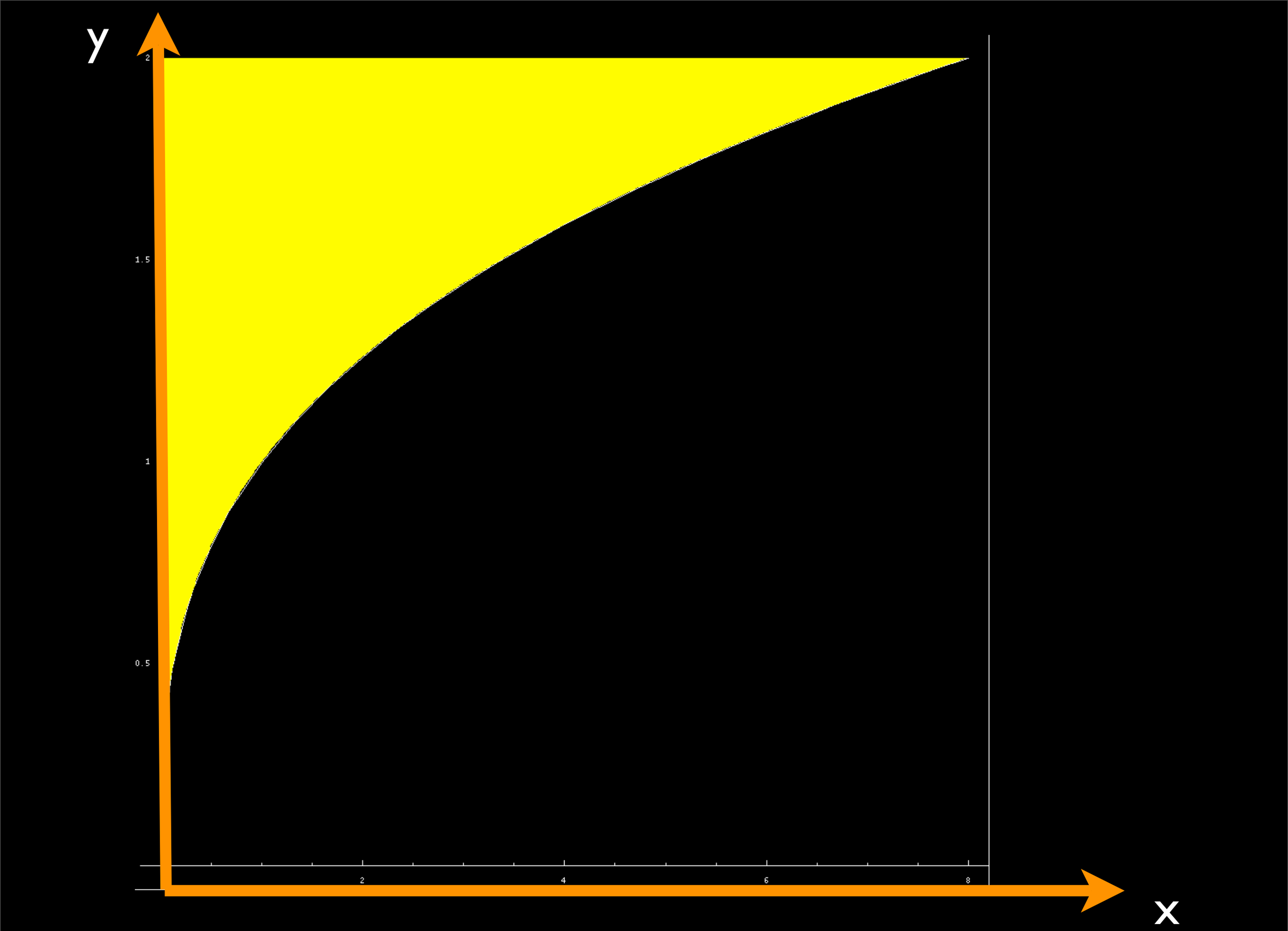
$$\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4+1}\, dy\, dx$$

Look at the inner integral:

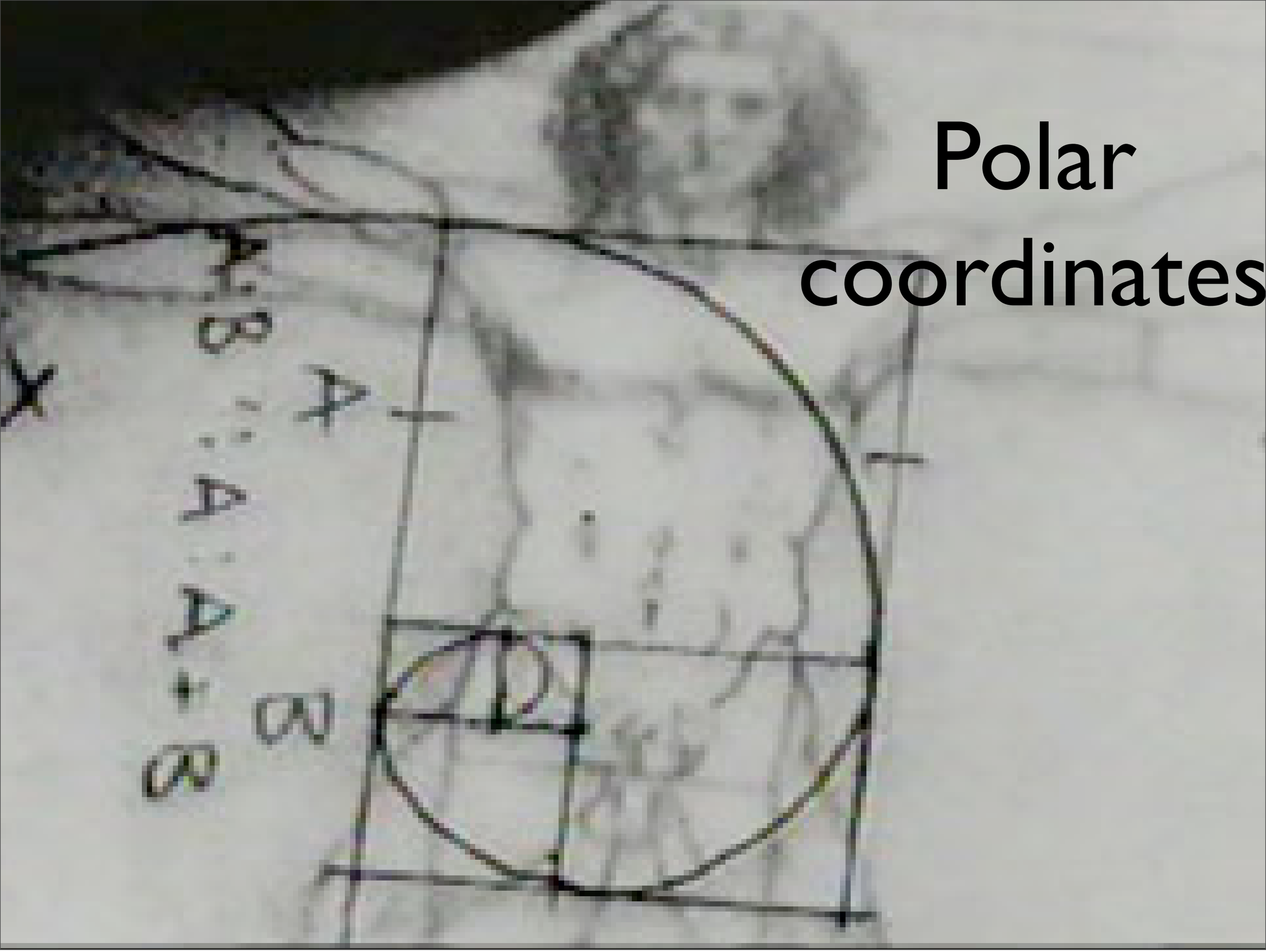
$$\int_0^8 \int_{x^{1/3}}^2 \frac{1}{y^4+1}\, dy\, dx$$

It does not look good.


If a double integral you
can not solve, the
integration order change
you must


y
2
1.5
1
0.5
2
4
6
8
x

If an integral solve you
can't, other coordinate
systems consider you
must



Polar coordinates

Polar coordinates

$$\int_0^{2\pi} \int_0^{r(\theta)} r \; dr \; d\theta$$

Example

$$r(\theta) = |\cos(n\theta)|$$

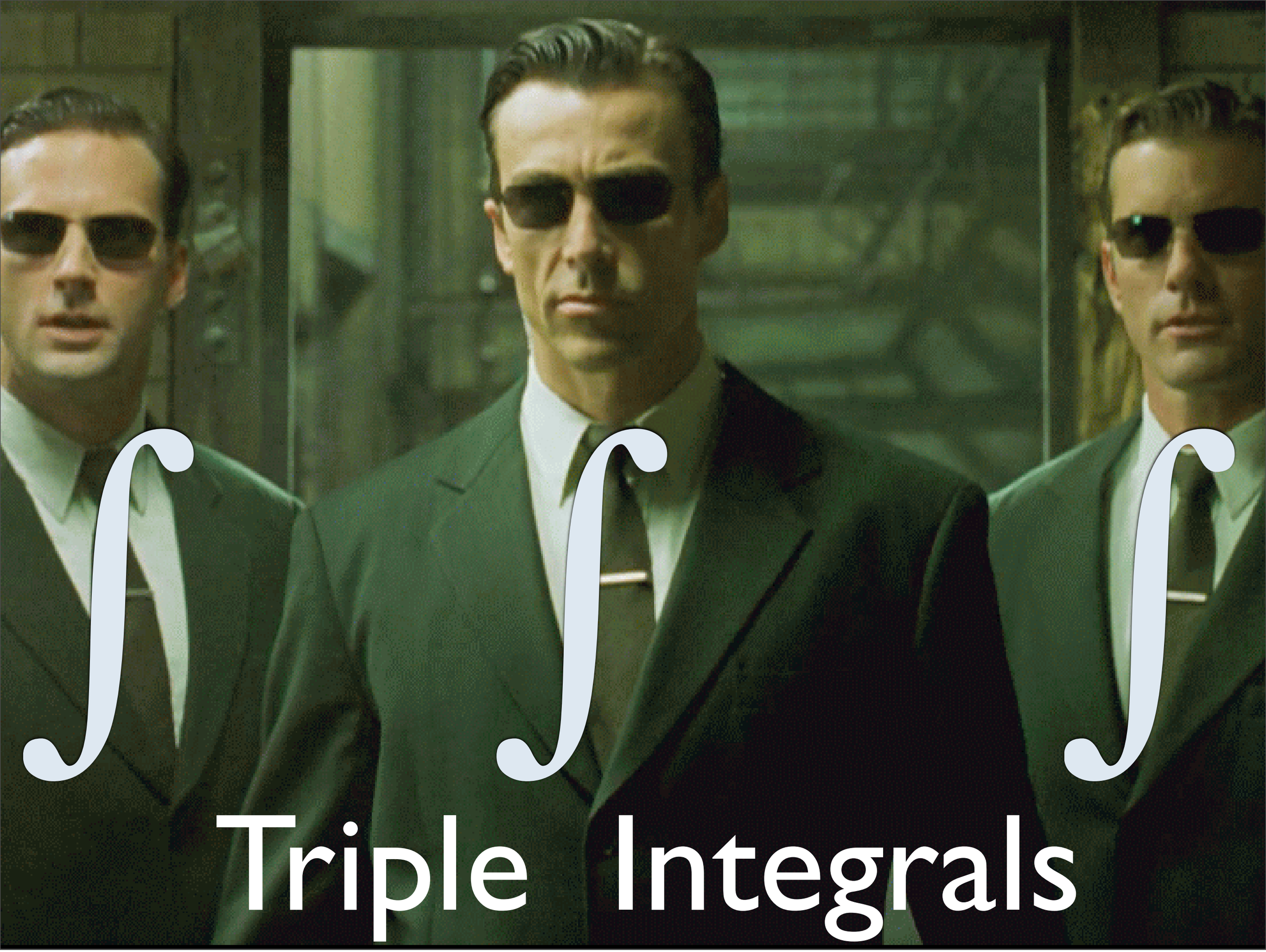
$$\int_0^{2\pi} \frac{\cos^2(n\theta)}{2} \, d\theta = \frac{\pi}{2}$$

Problem: What is

$$\int_0^4 \int_0^{\sqrt{16-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$$

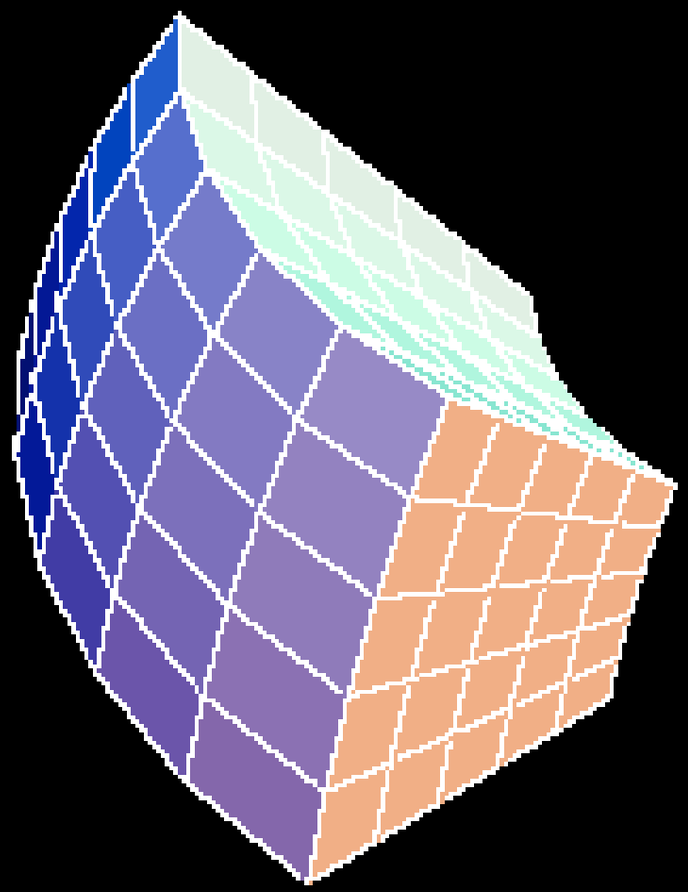
This smells like Polar coordinates!

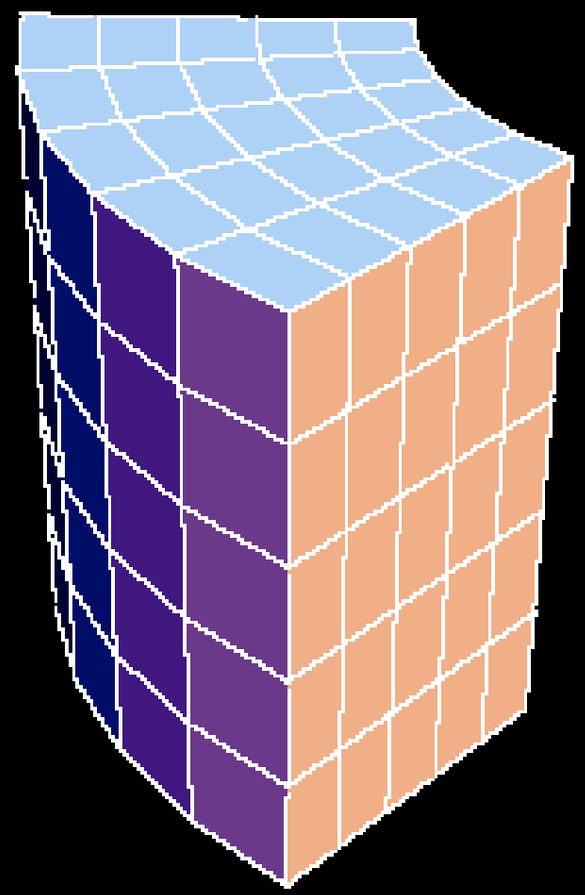
$$\int_0^4 \int_0^{\sqrt{16-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$$



Triple Integrals

Spherical/Cylindrical Coordinates

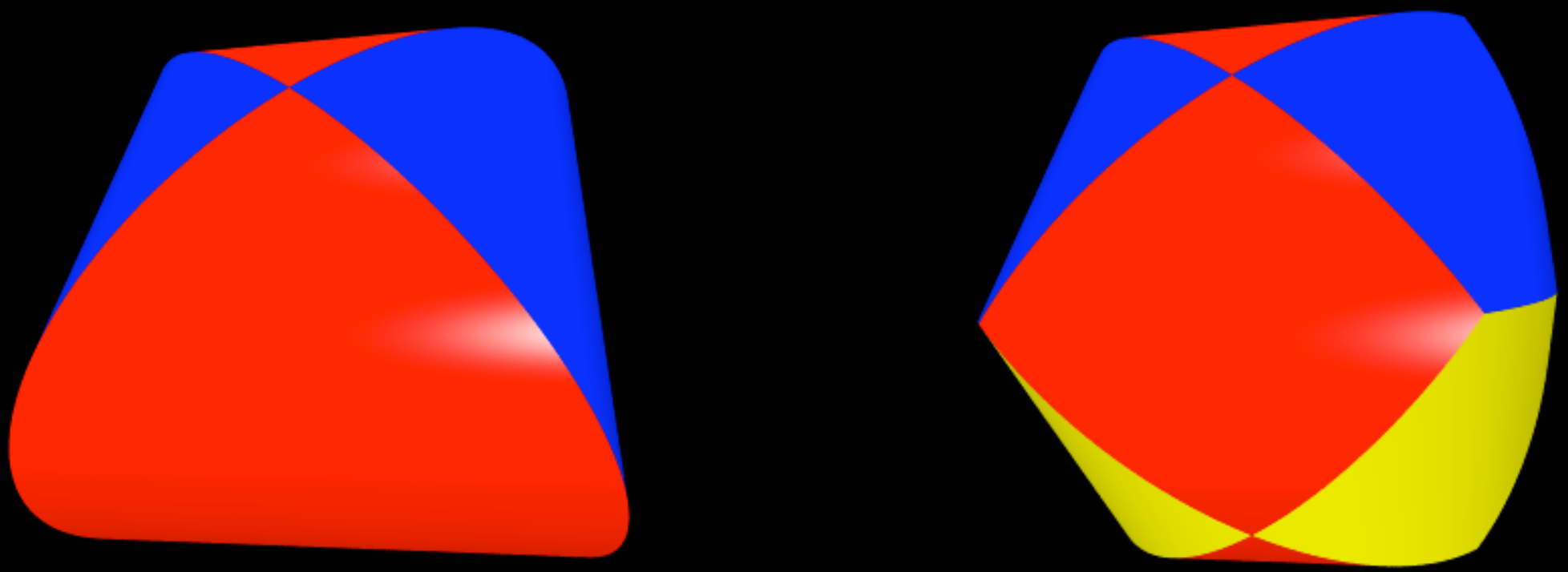




Remember the integration factors? You should know them by heart.

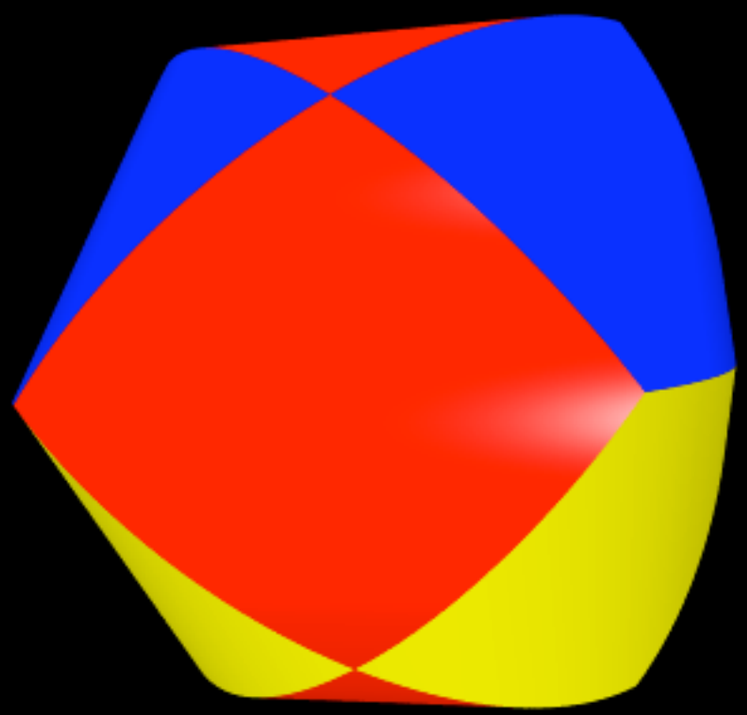
Triple Integrals

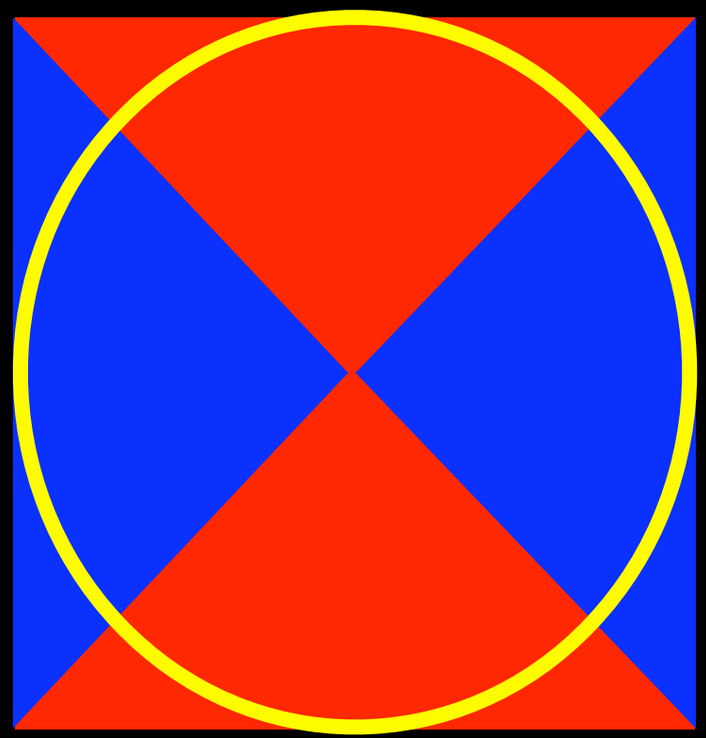
Example: Volume computation



$$\int\int\int_E 1 \; dV = \text{Vol(E)}$$

Make good pictures!



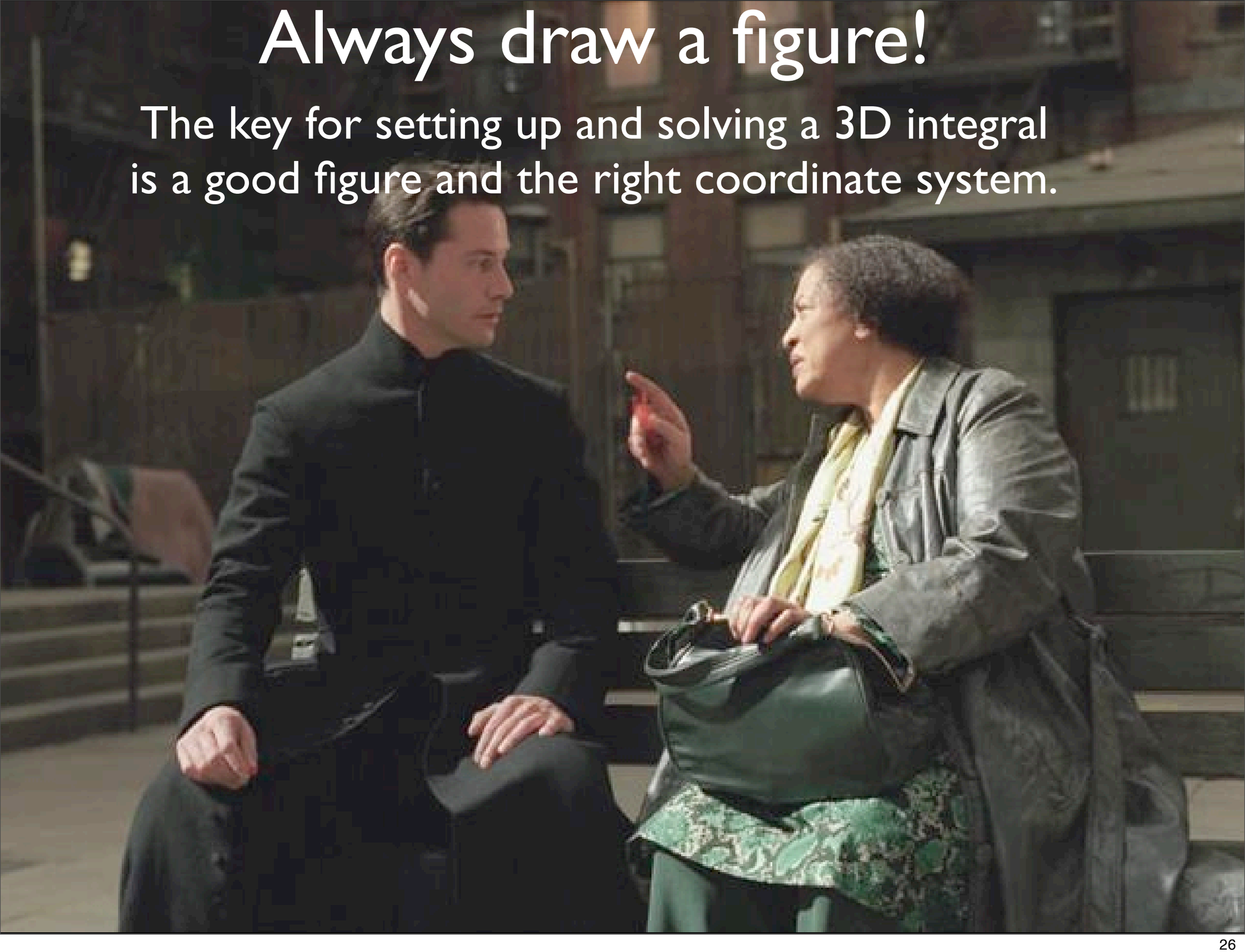


Example: Volume of the intersection of the three cylinders:

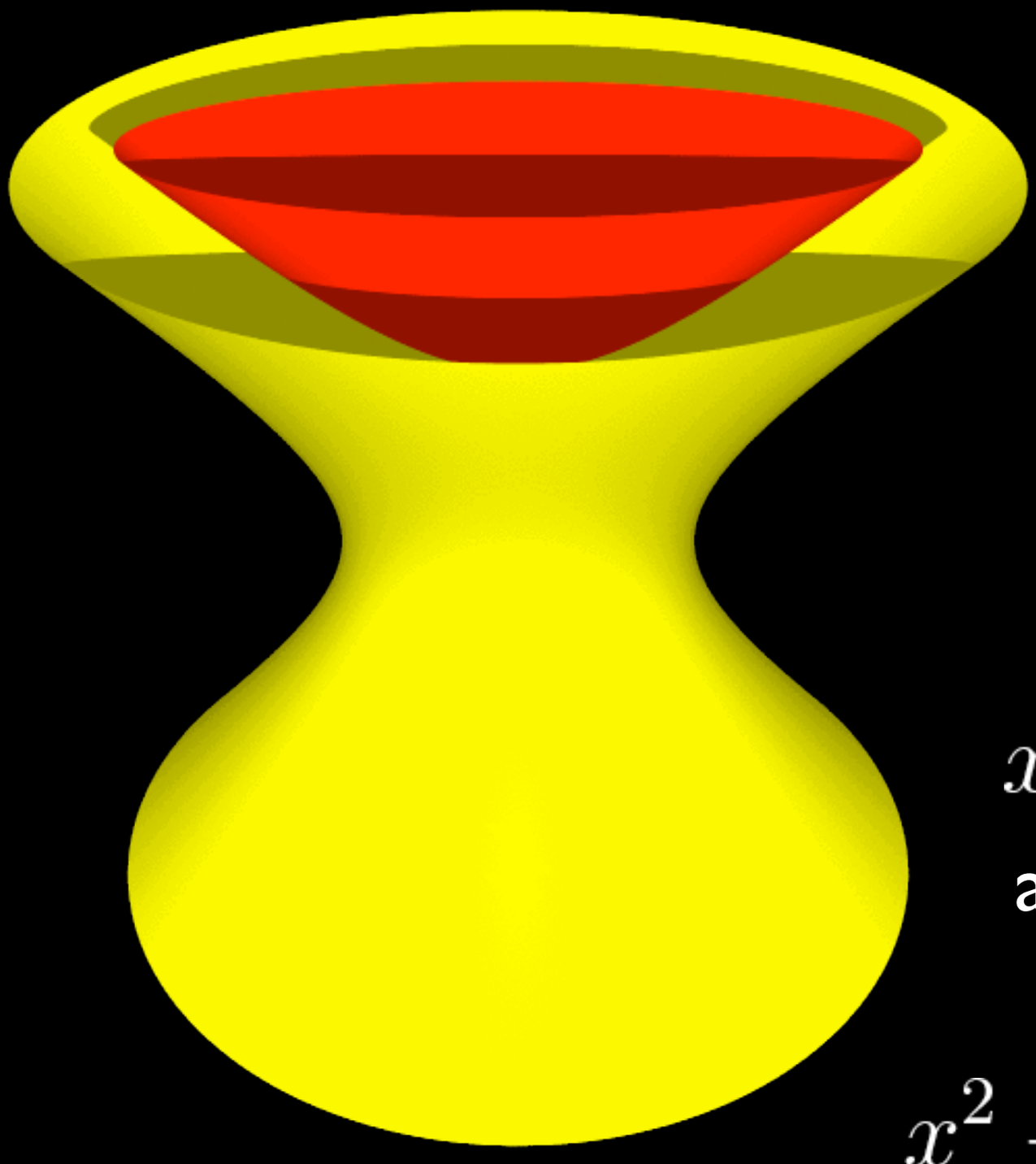
$$8\int_{-\pi/4}^{\pi/4}\int_{0}^{1}\sqrt{1-r^2\sin^2(t)}r\;drdt = -\frac{16}{3}+8\sqrt{2}$$

Always draw a figure!

The key for setting up and solving a 3D integral is a good figure and the right coordinate system.



Problem:



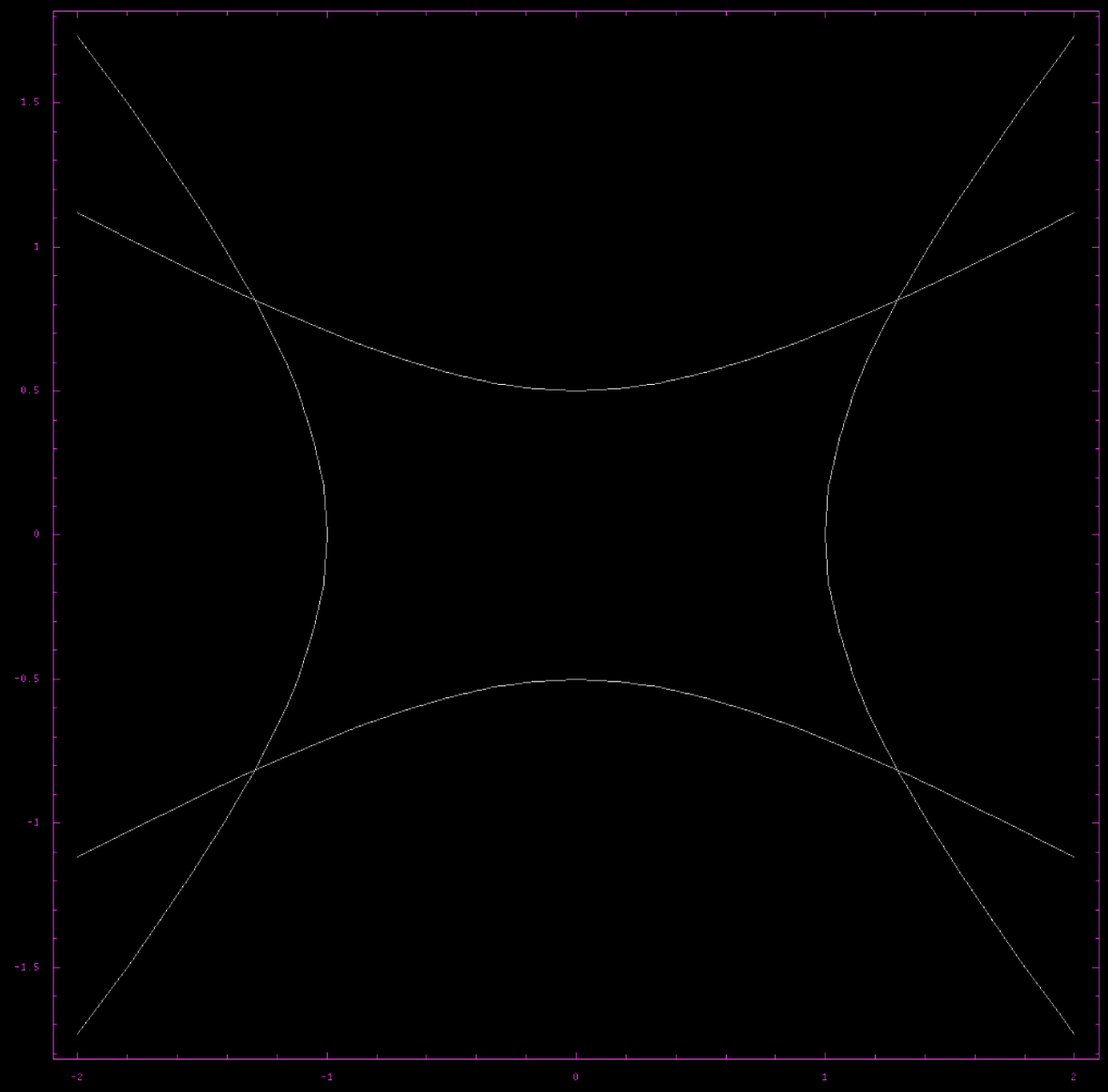
Find the volume of the solid bound by the one sheeted hyperboloid

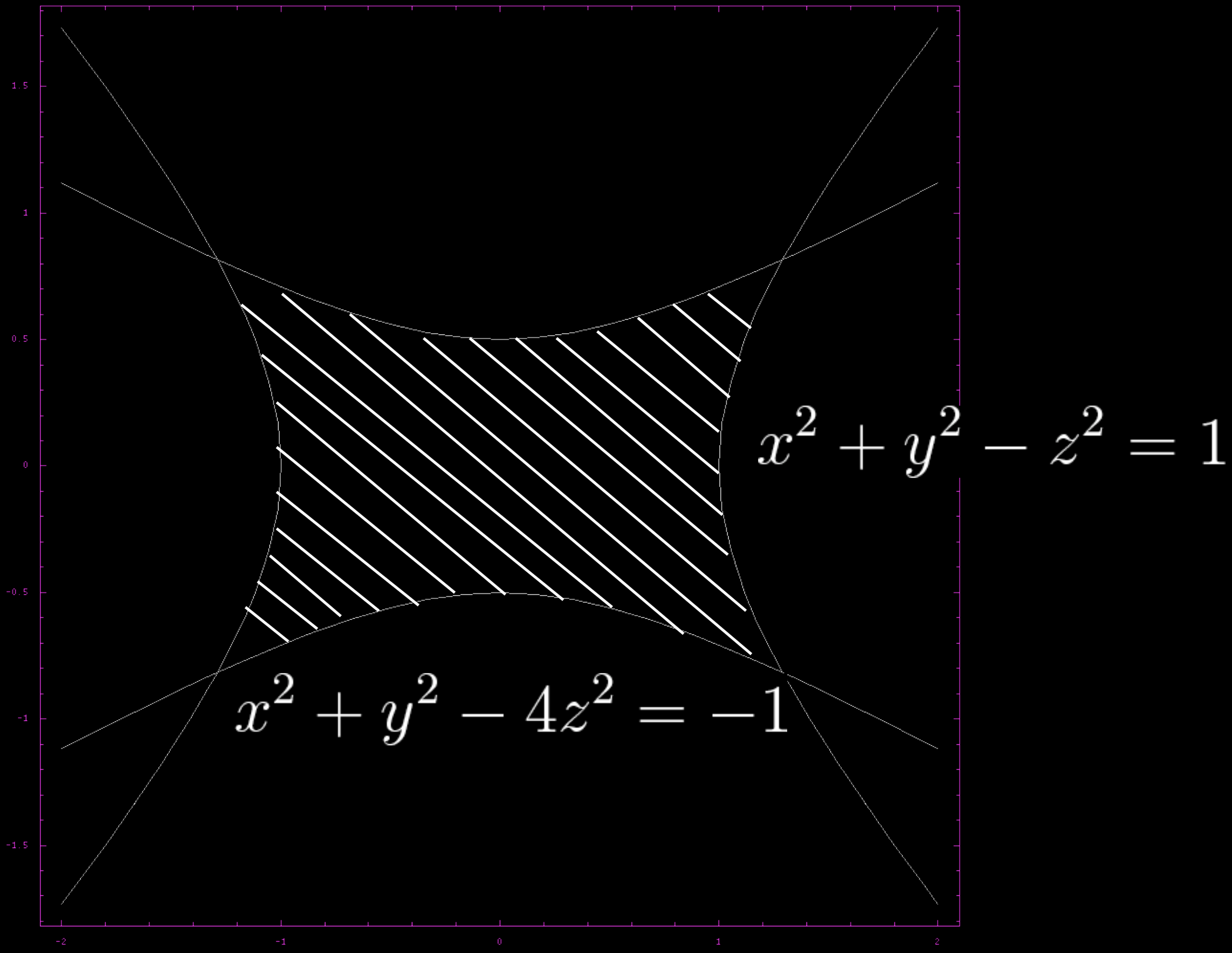
$$x^2 + y^2 - z^2 = 1$$

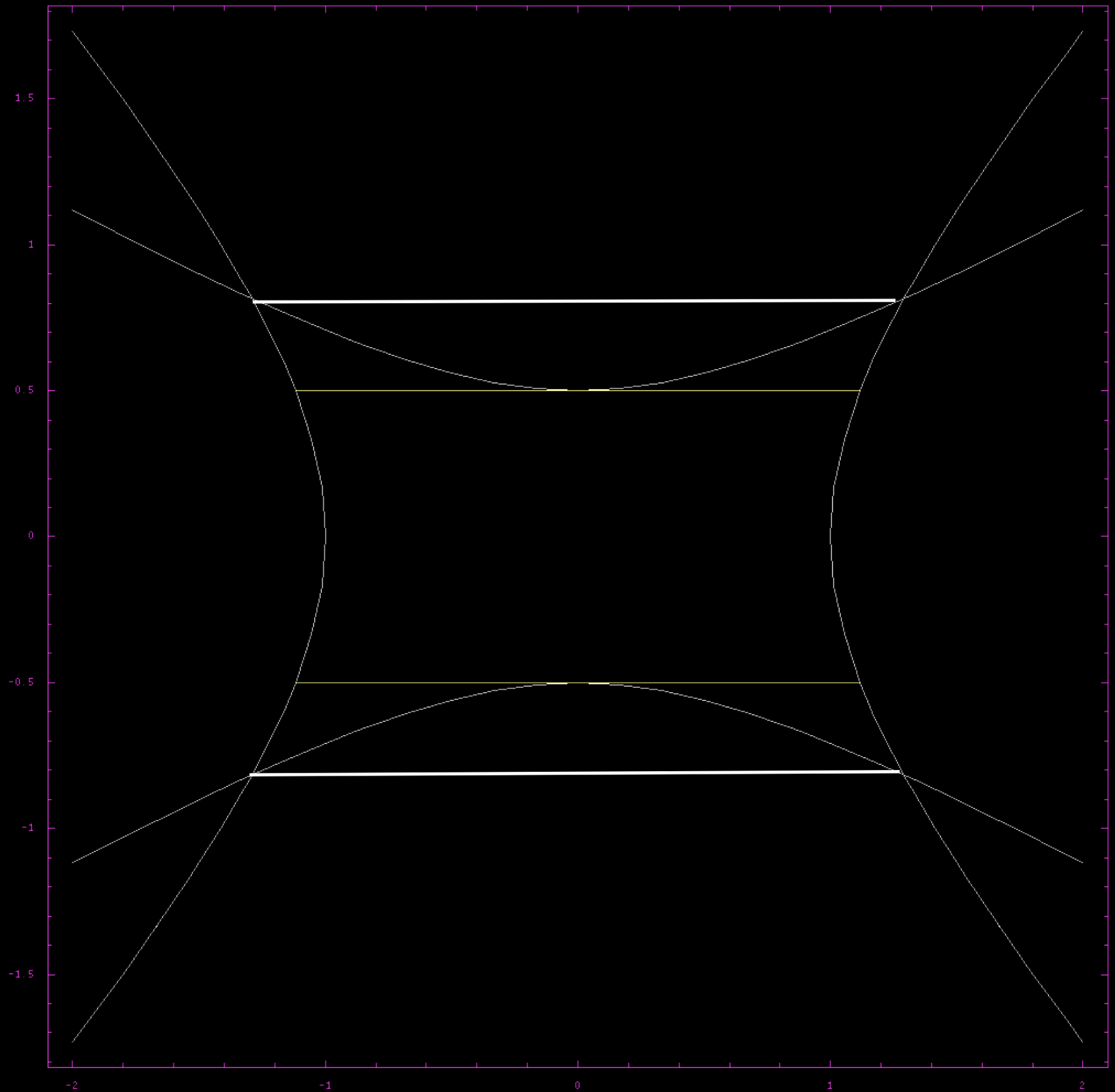
and the two sheeted hyperboloid

$$x^2 + y^2 - 4z^2 = -1$$

It is crucial to have a good picture




1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2
$x^2 + y^2 - z^2 = 1$
$x^2 + y^2 - 4z^2 = -1$

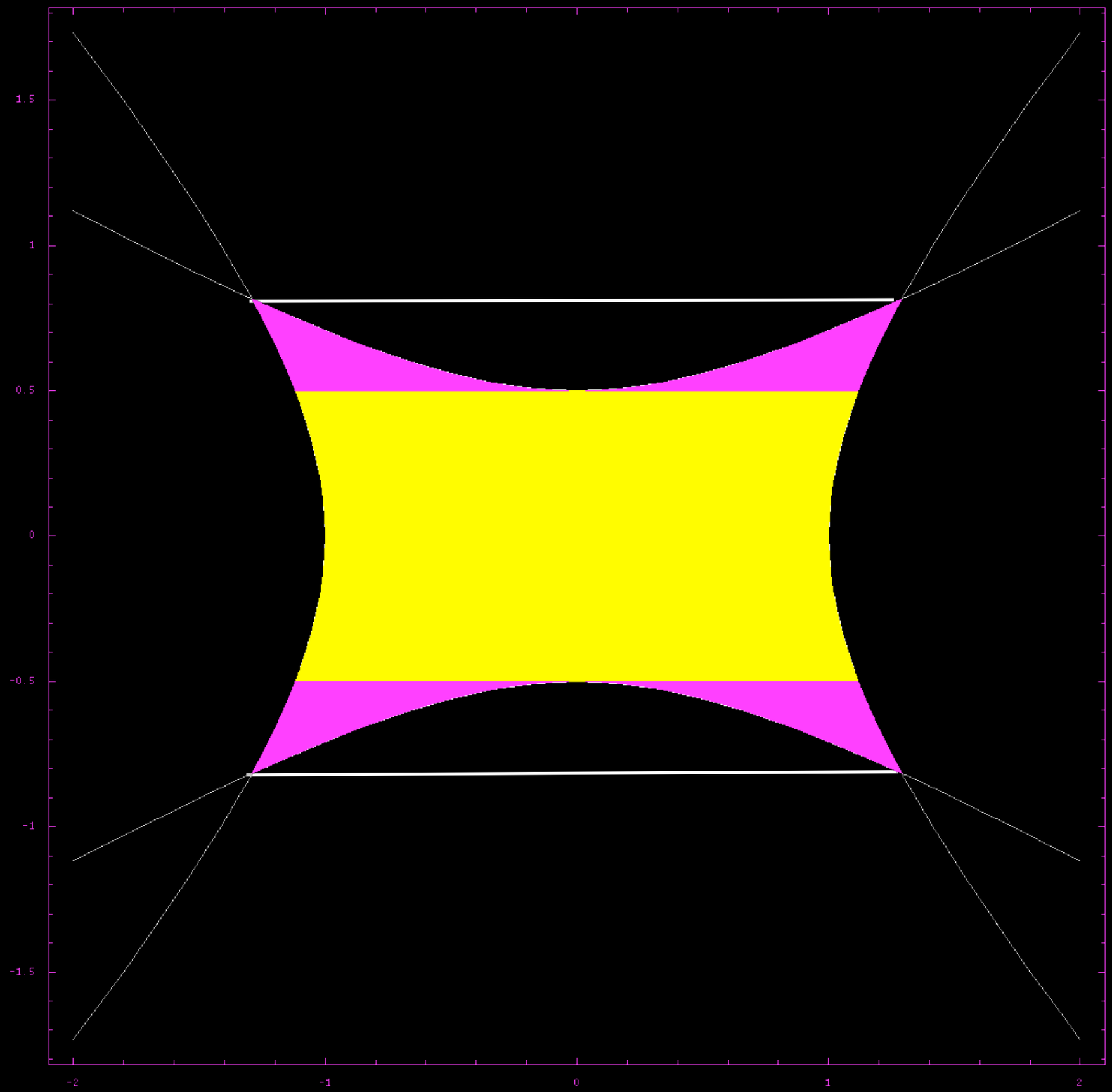


$\sqrt{2/3}$

1/2

-1/2

$-\sqrt{2/3}$



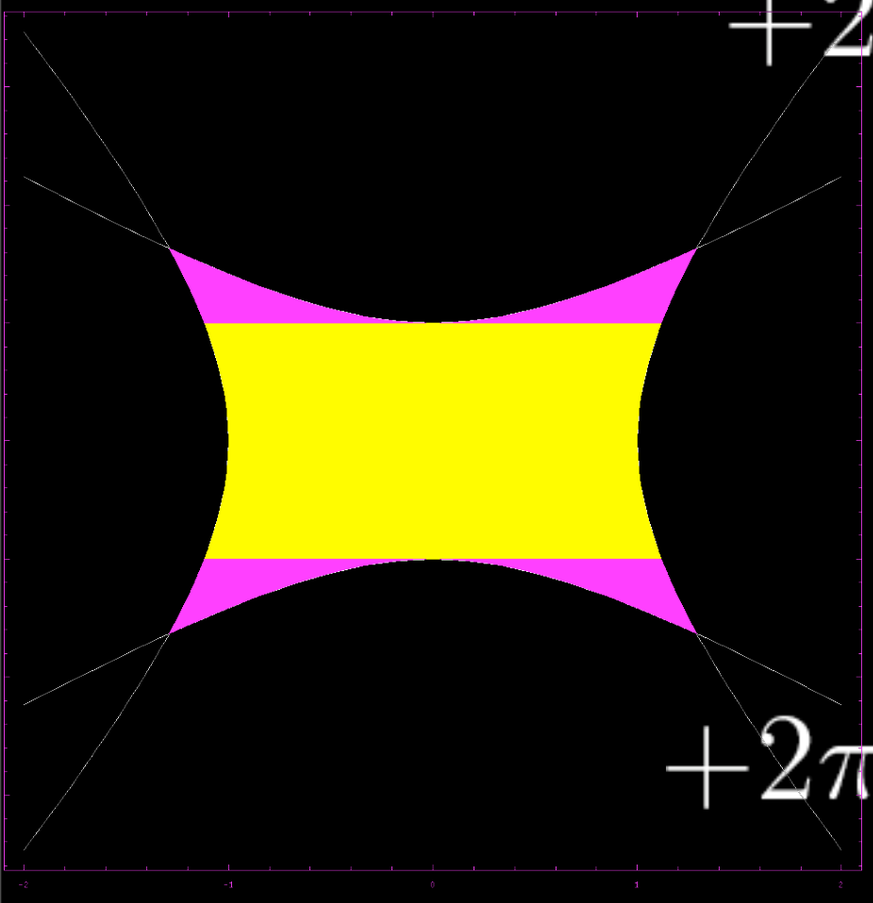
$\sqrt{2/3}$

1/2

-1/2

$-\sqrt{2/3}$

$$2\int_0^{1/2}\int_0^{\sqrt{1+z^2}}\int_0^{2\pi} r\ d\theta\ dr\ dz$$



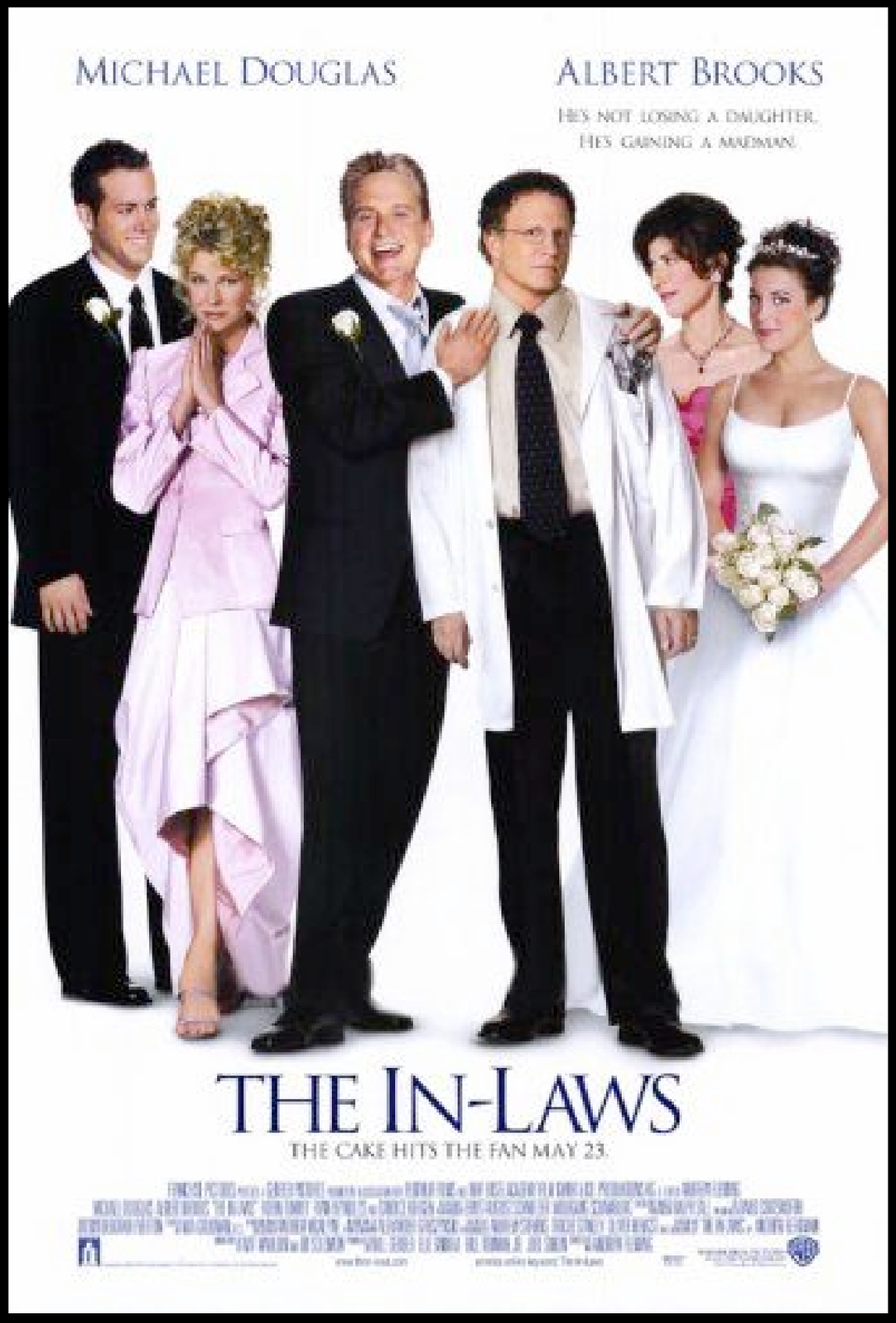
$$+2\int_{1/2}^{\sqrt{2/3}}\int_{\sqrt{-1+4z^2}}^{\sqrt{1-z^2}}\int_0^{2\pi} r\ d\theta\ dr\ dz$$

$$= 2\int_0^{1/2}\pi(1+z^2)\ dz$$

$$+2\pi\int_{1/2}^{\sqrt{2/3}}(1+z^2)-(-1+4z^2)\ dz$$

$$=\pi\ 13/12 \quad + (32\sqrt{6}\ -63)/36\ \pi$$

Quiz coming up



Win some DVD

The inlaws

Let V be the integral of

$f(x,y,z) = x+y+z+1/(7\pi)$

over the ball with radius 1/2 .What is 1/V?

Answer:

We don't know the ultimate question. But we have the ultimate advise

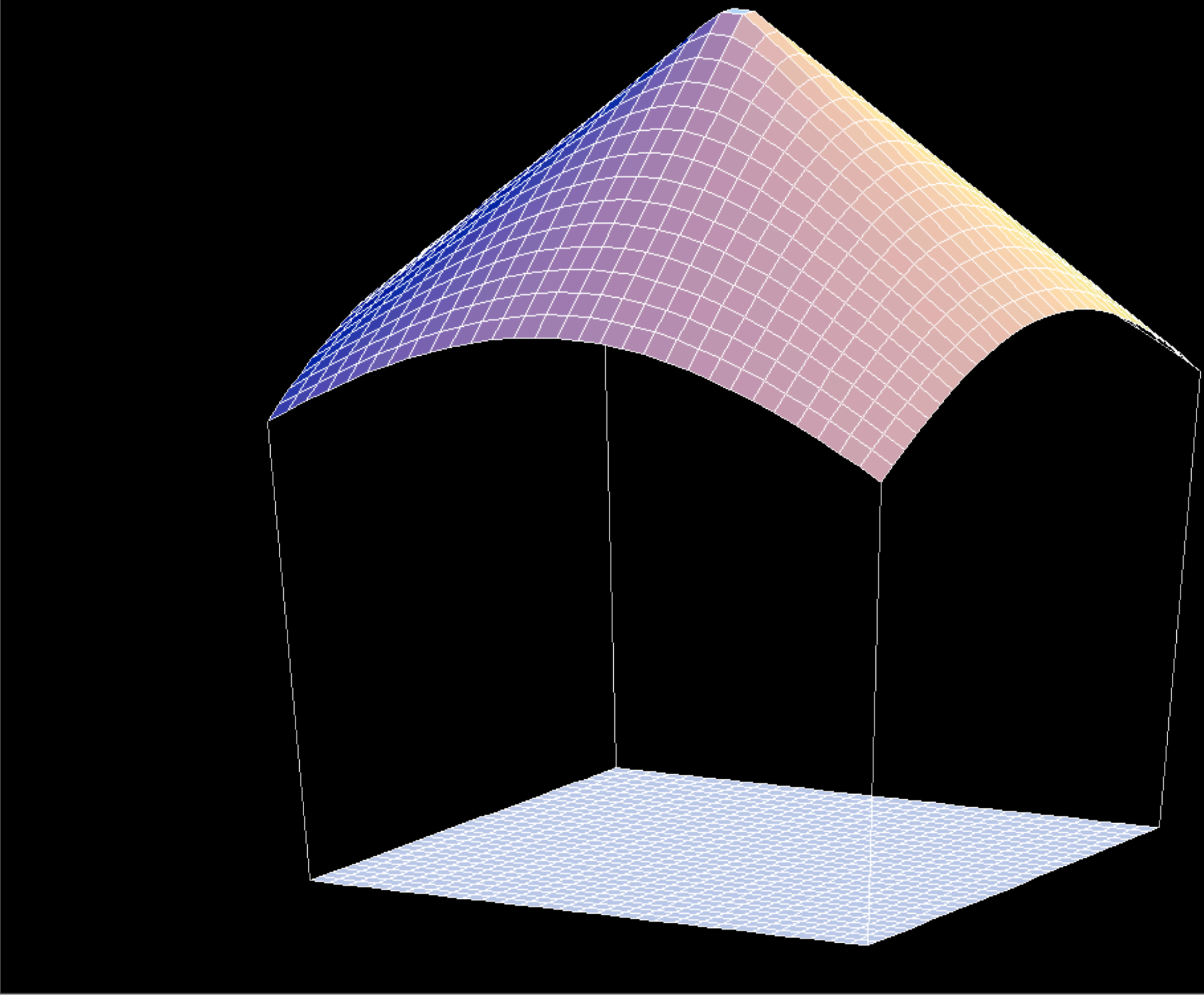
- Make a good picture
- Consider other coordinate systems
- Switch order of integration if necessary

It is important, that if you have a problem, like computing the volume of

$-1<x<1,-1<y<1,0<z<\sqrt{x^2+y^2}$

that you

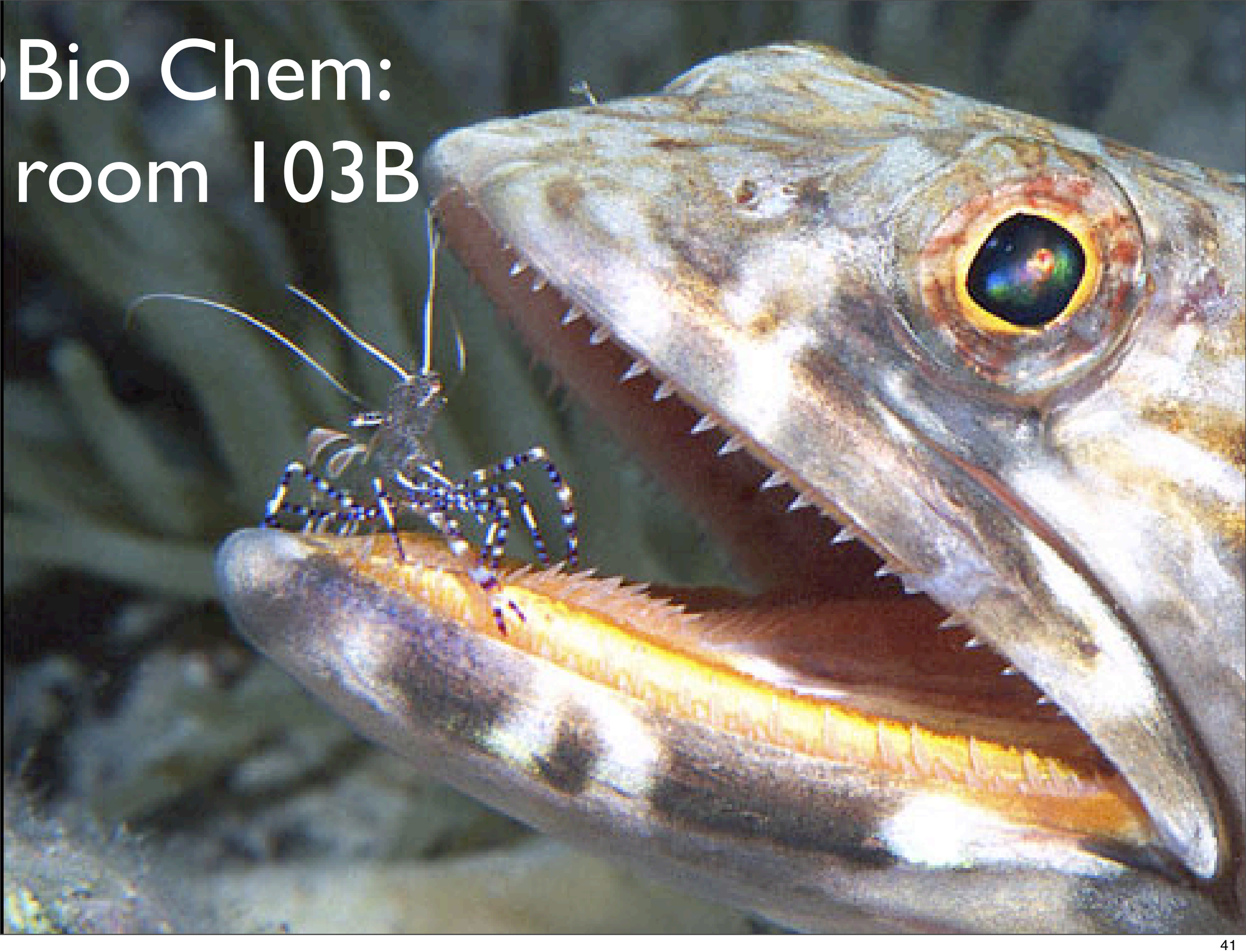
Visualize it!

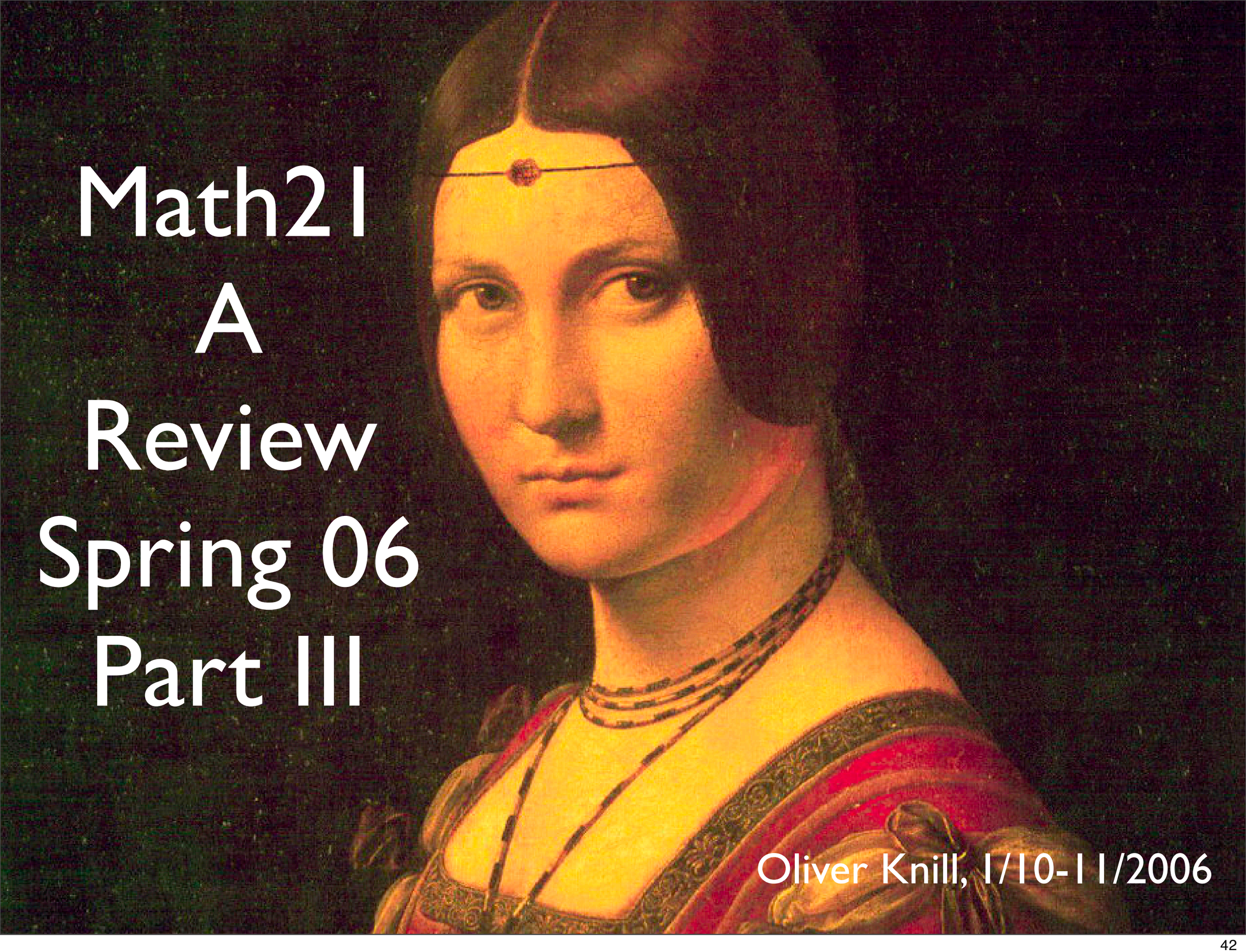


Short 5 min Break!

- The bio chem section will continue at 3PM in room 103 B.

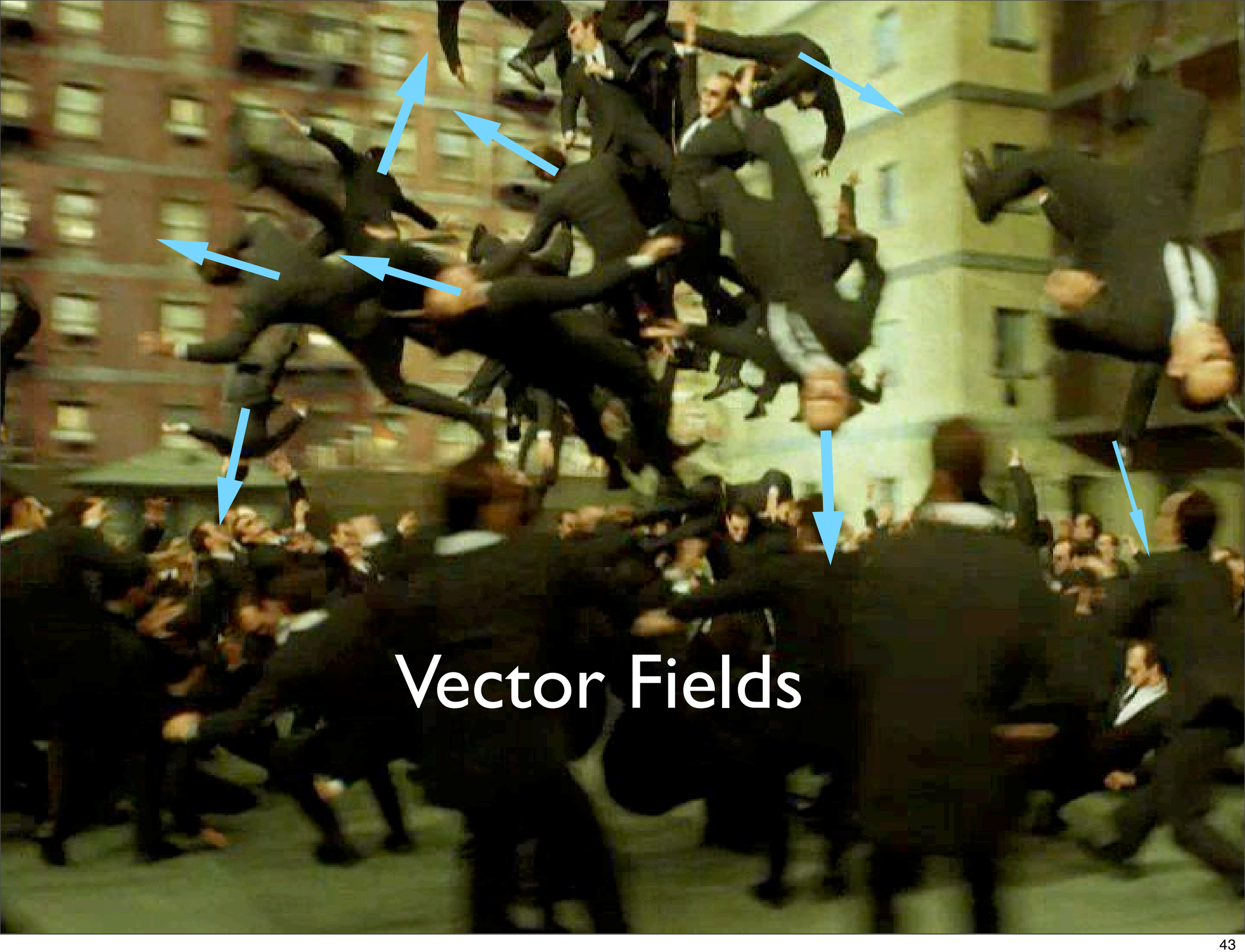
Bio Chem: room 103B



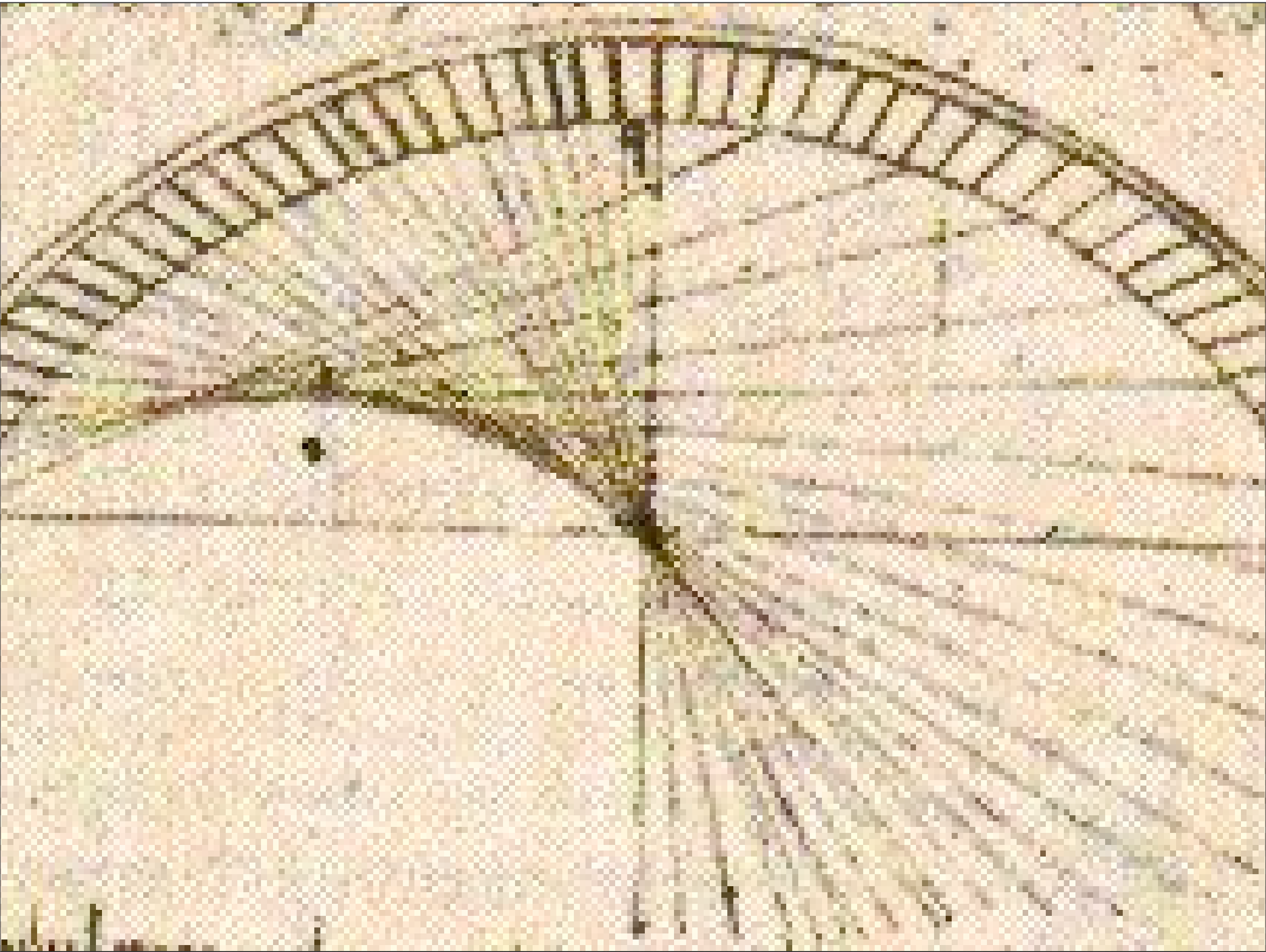


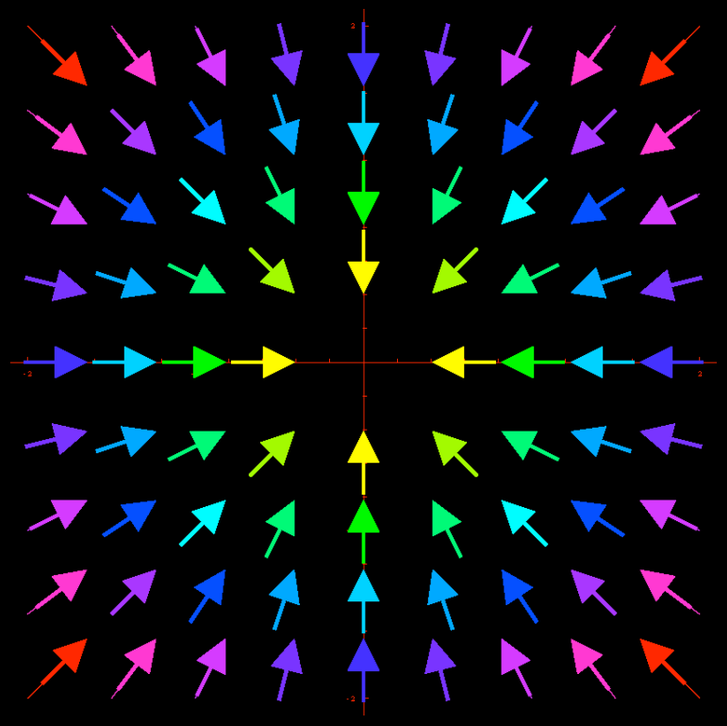
Math21 A Review Spring 06 Part III

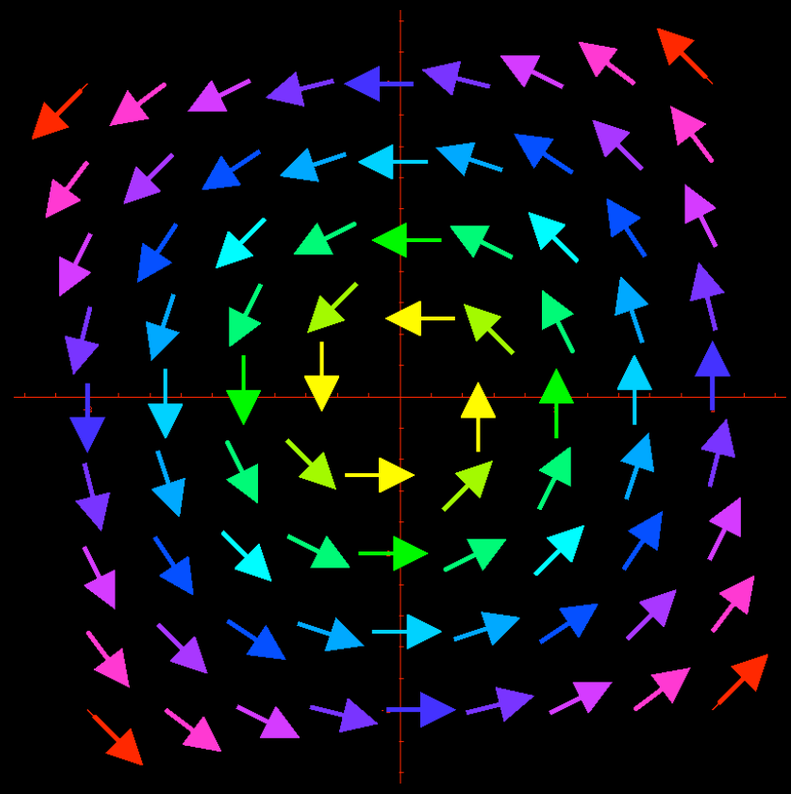
Oliver Knill, 1/10-11/2006


Vector Fields

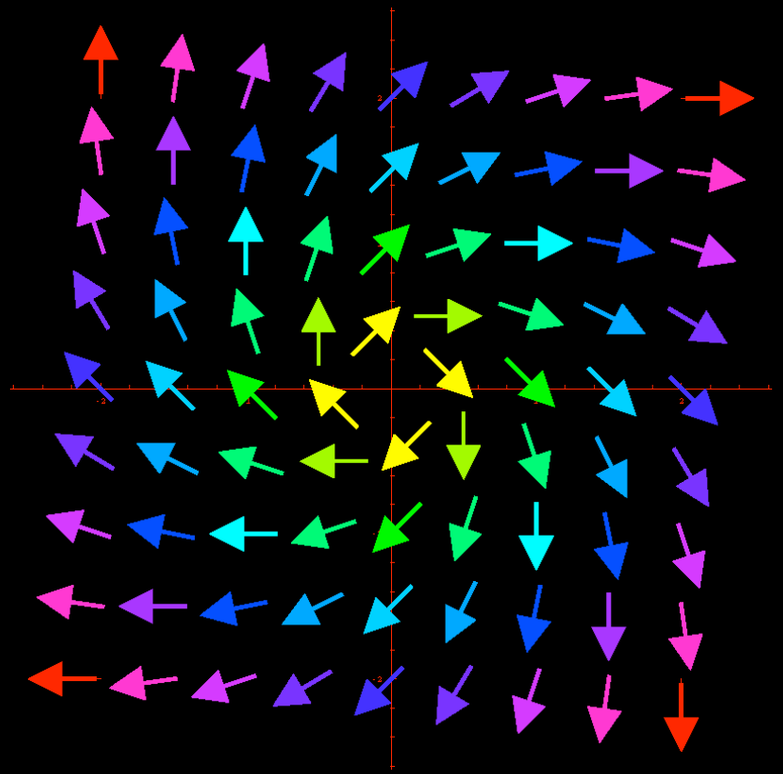
Vector Fields

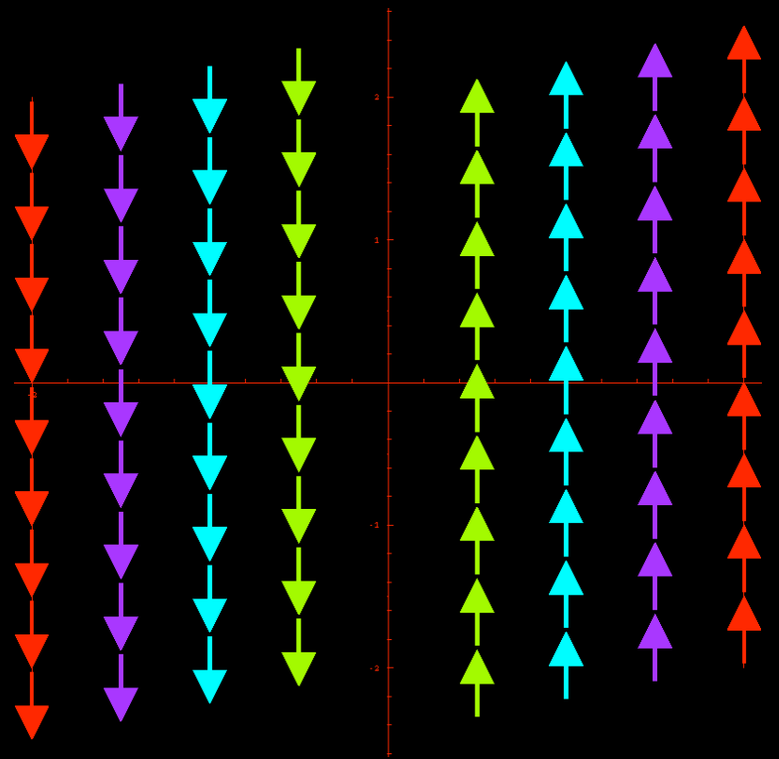


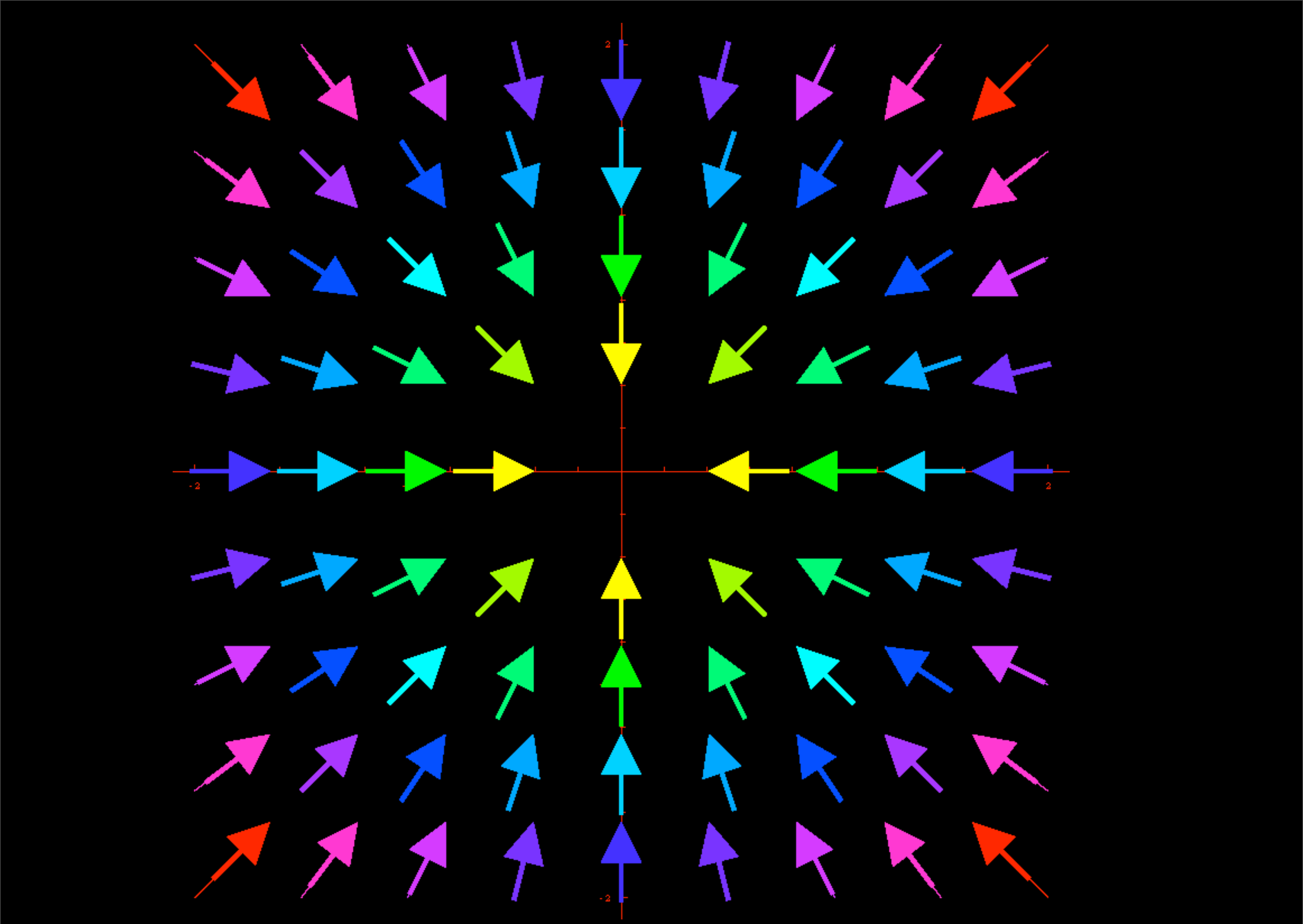


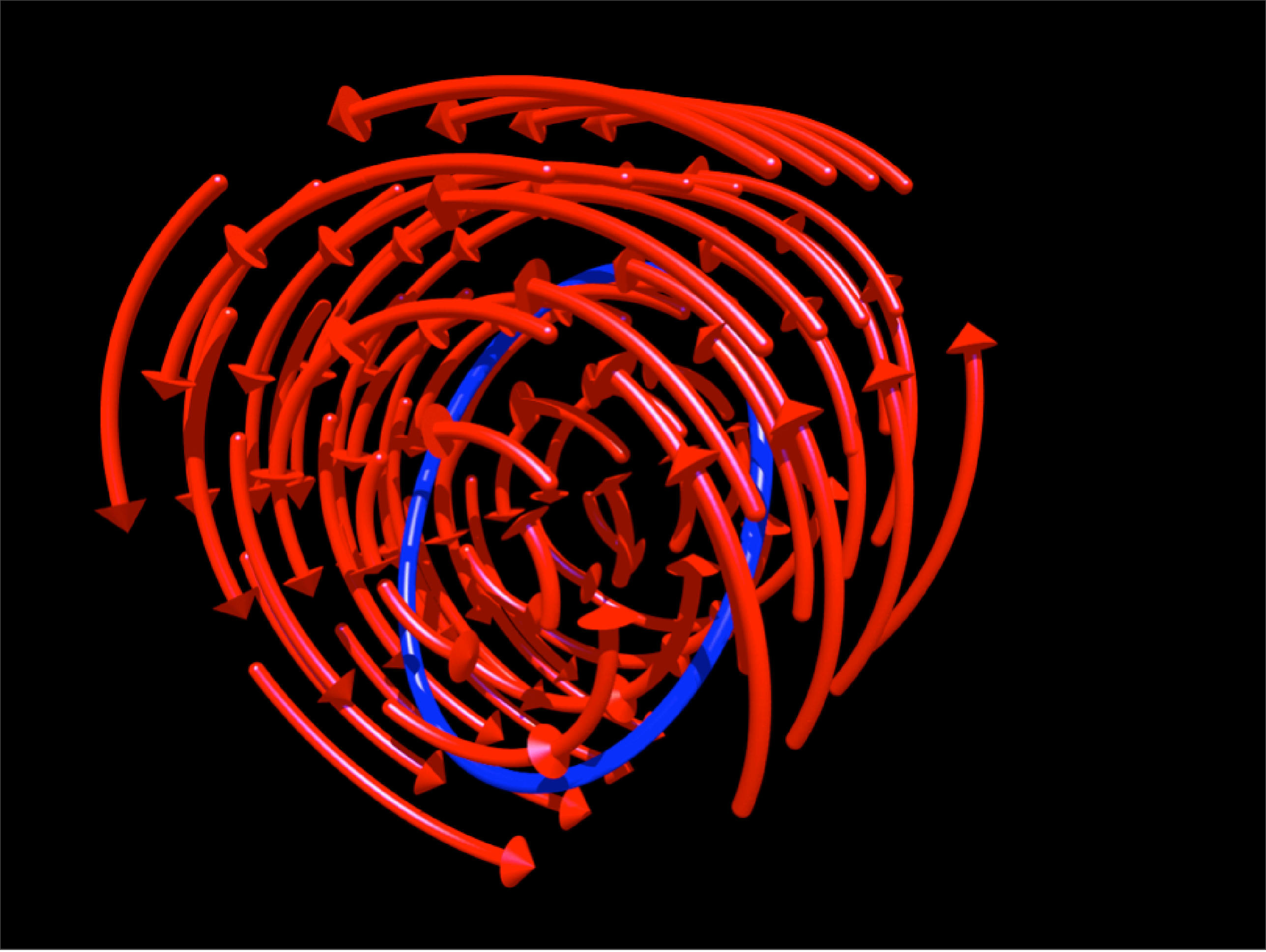


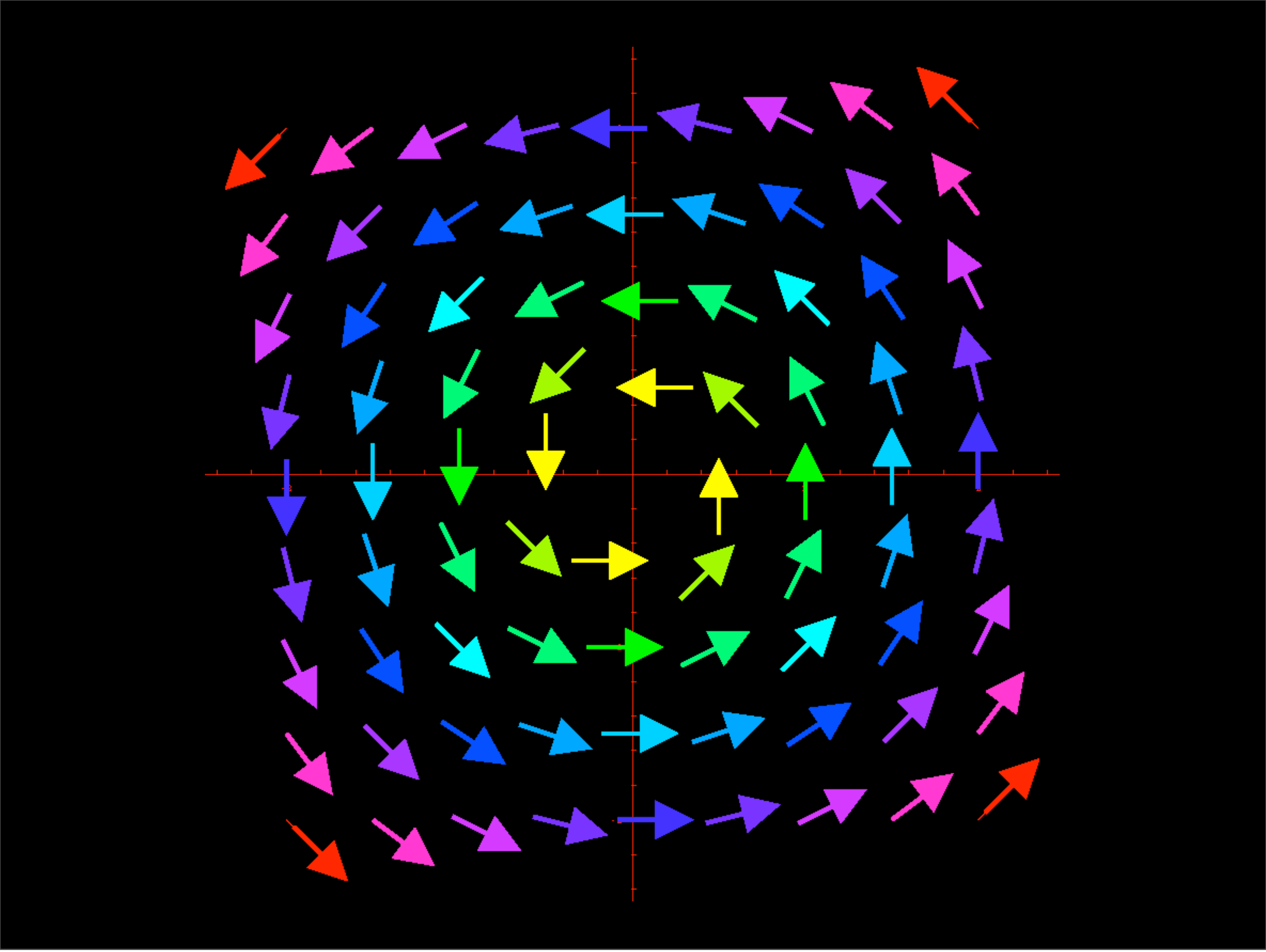
Vector fields

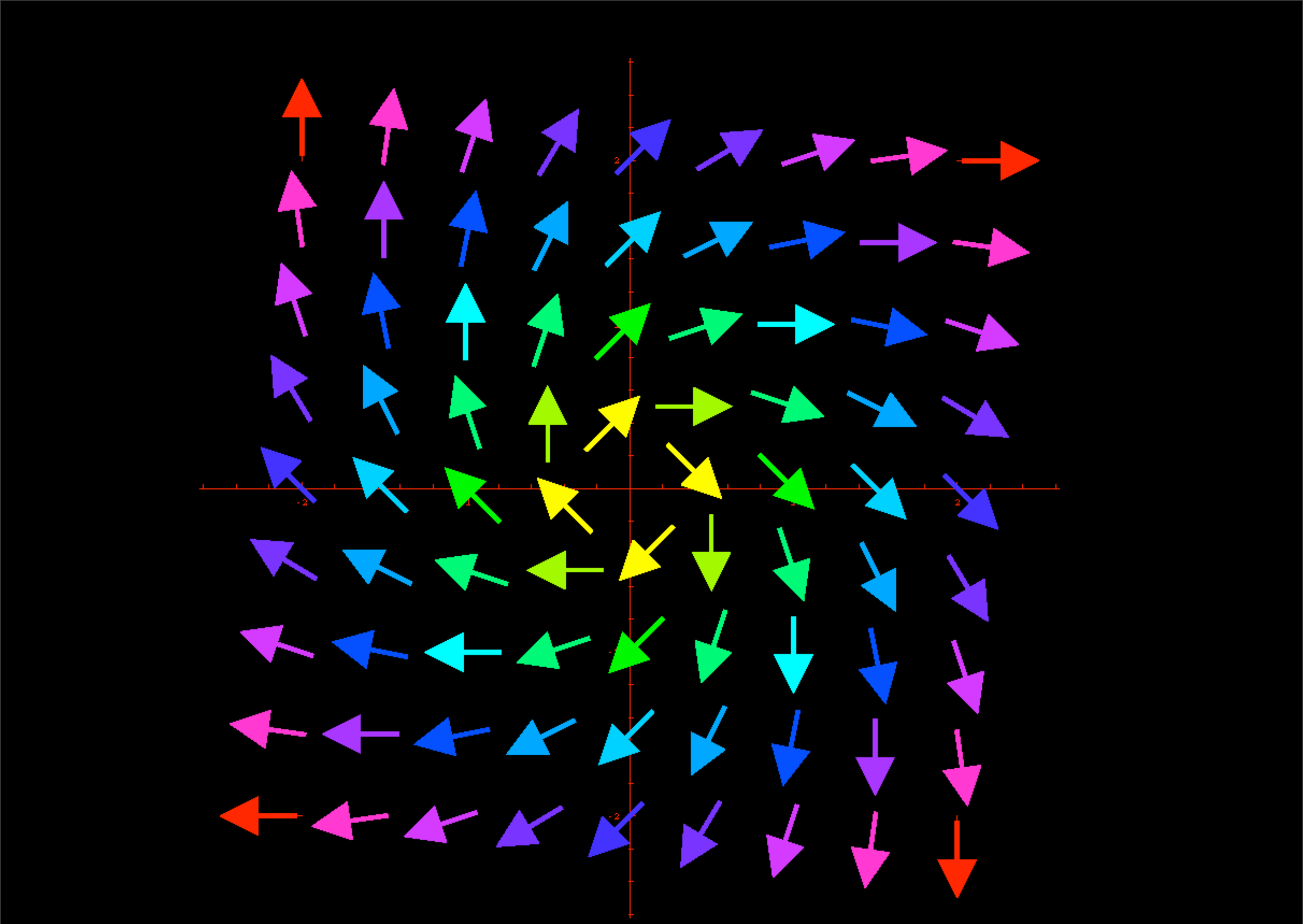


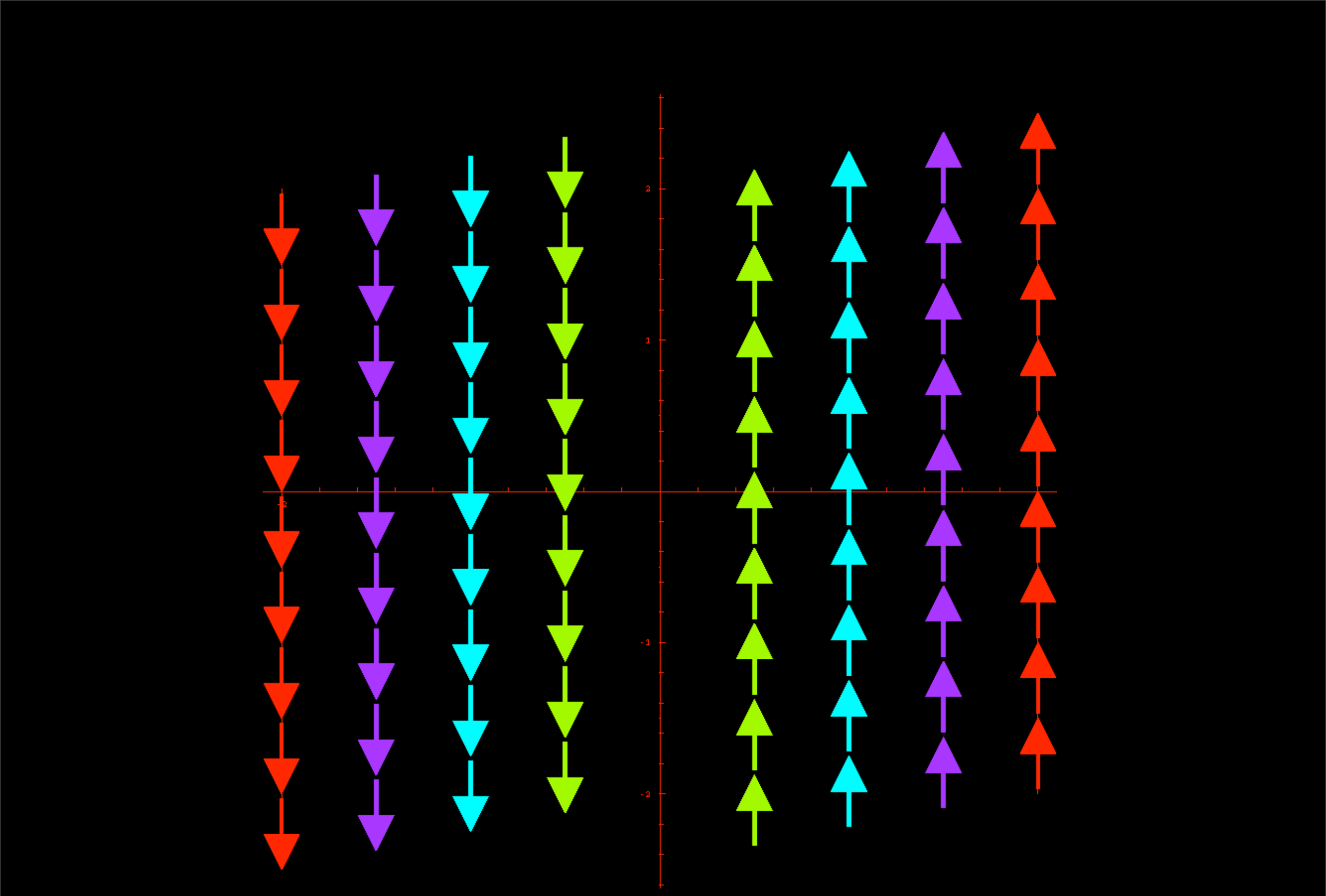



2
-2
2
-2





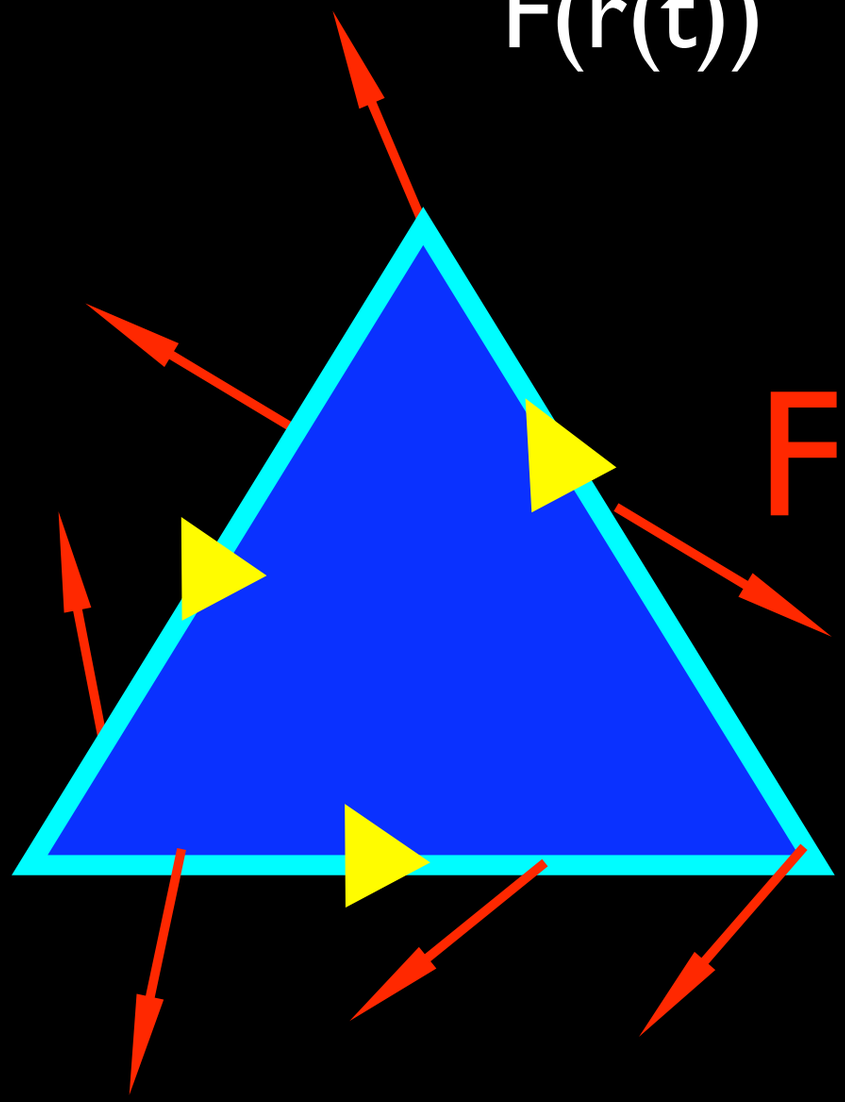



2
1
-1
-2

Lineintegrals

F(r(t)) "Force at point r(t)"

$$F(r(t)) \cdot r'(t)$$



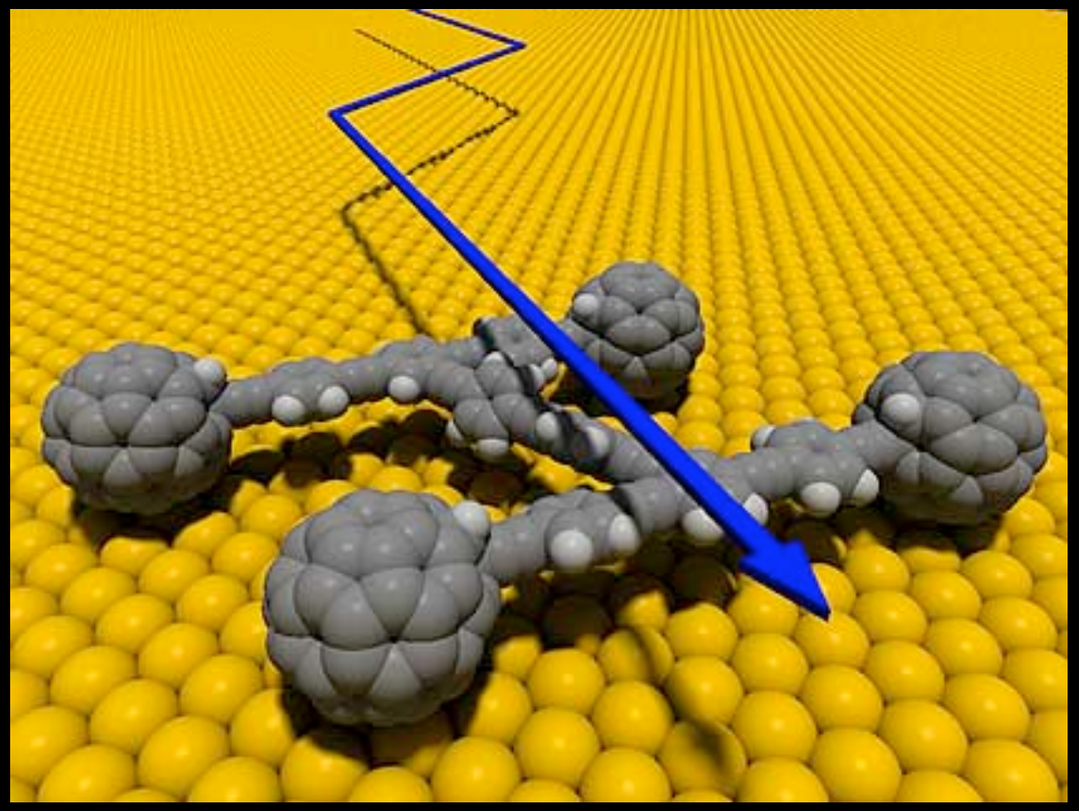
F

= "Power"

Integrating this over the time interval gives an energy.

Is this line integral positive or negative?

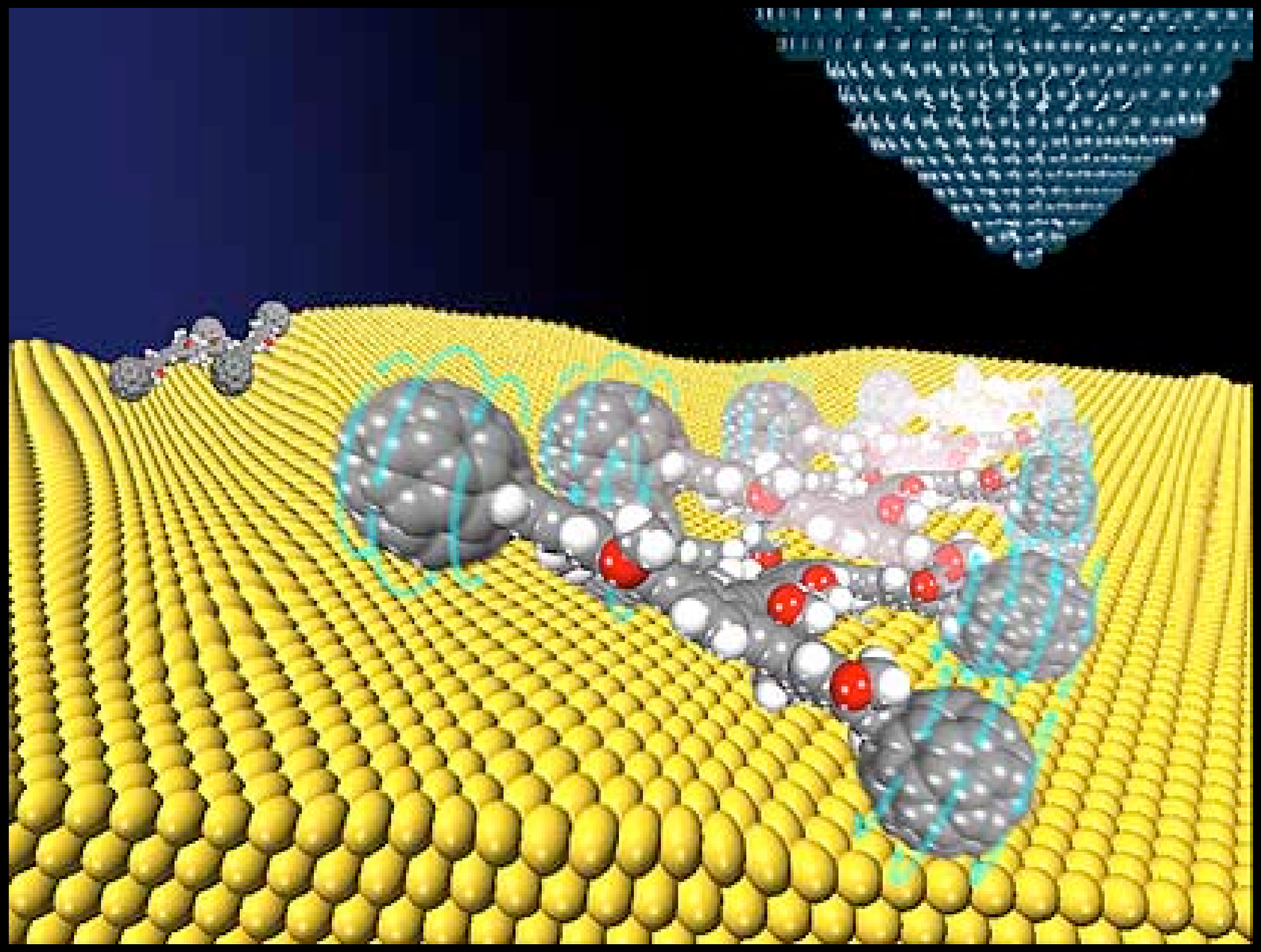
Nano car problem



At Rice university, one has build the first nanocar, it is 20’000 times smaller than the thickness of a hair and made of one molecule. The “street” is a gold plate which is heated up.

Nano car problem

The car is exposed to a force field F(x,y,z) = (y z ,x,z) from the surface and pushed along a path r(t) = (t,cos(t),sin(t)), where t goes from 0 to pi. What work is done on the car?



What do you think,
when you see this picture?

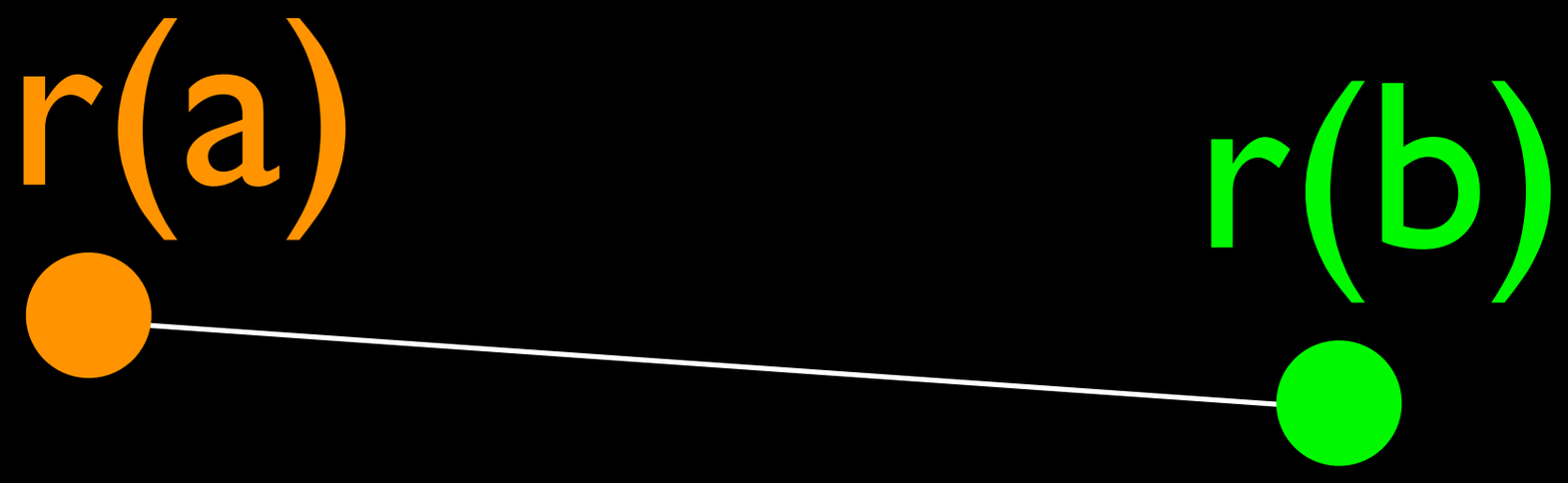
The Fundamental theorem of Lineintegrals of course!

f(r(a))

f(r(b))

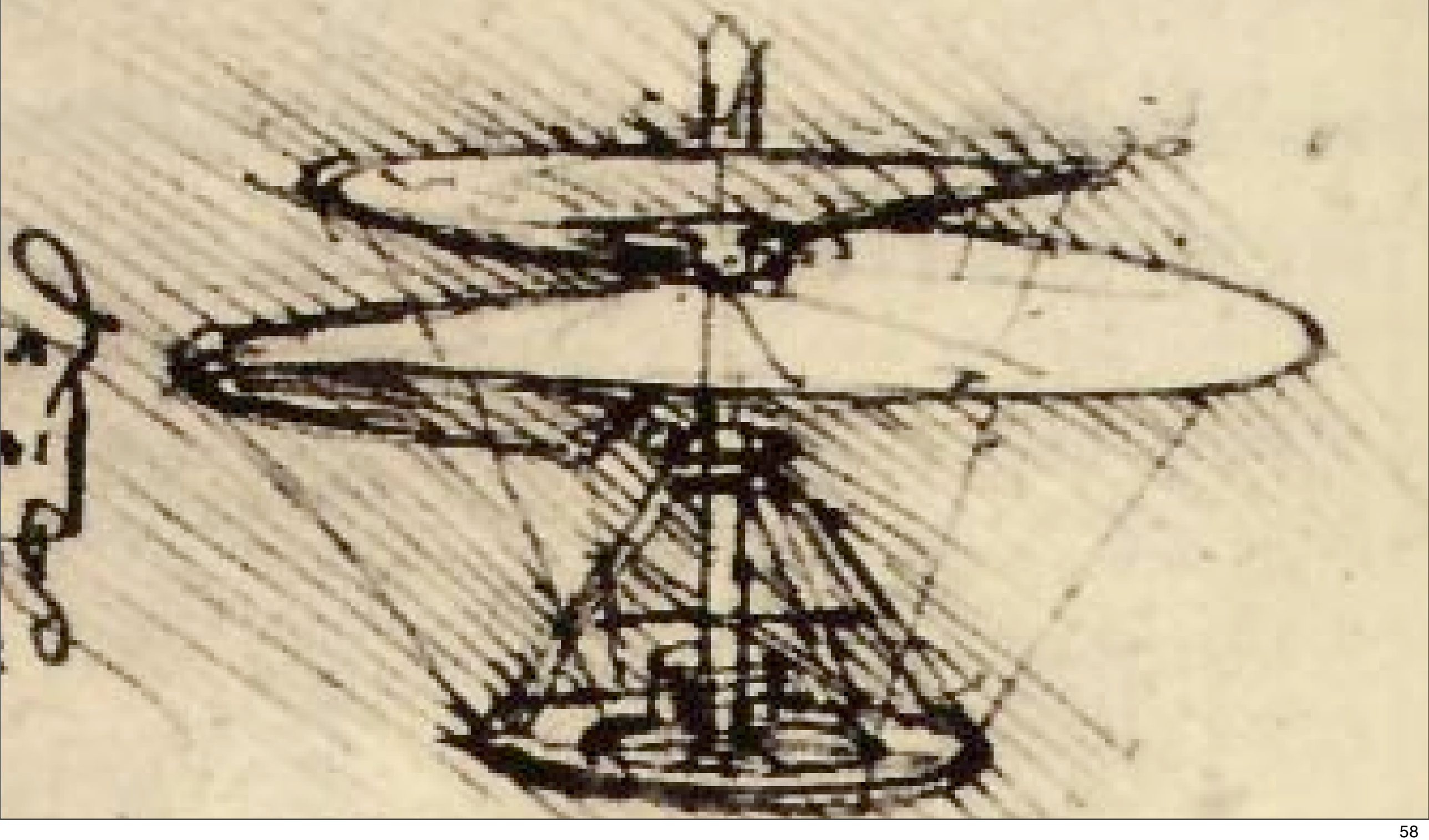
$$\int_\gamma \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Fundamental theorem of Lineintegrals



$$\int_\gamma \nabla f \cdot dr = f(r(b)) - f(r(a))$$

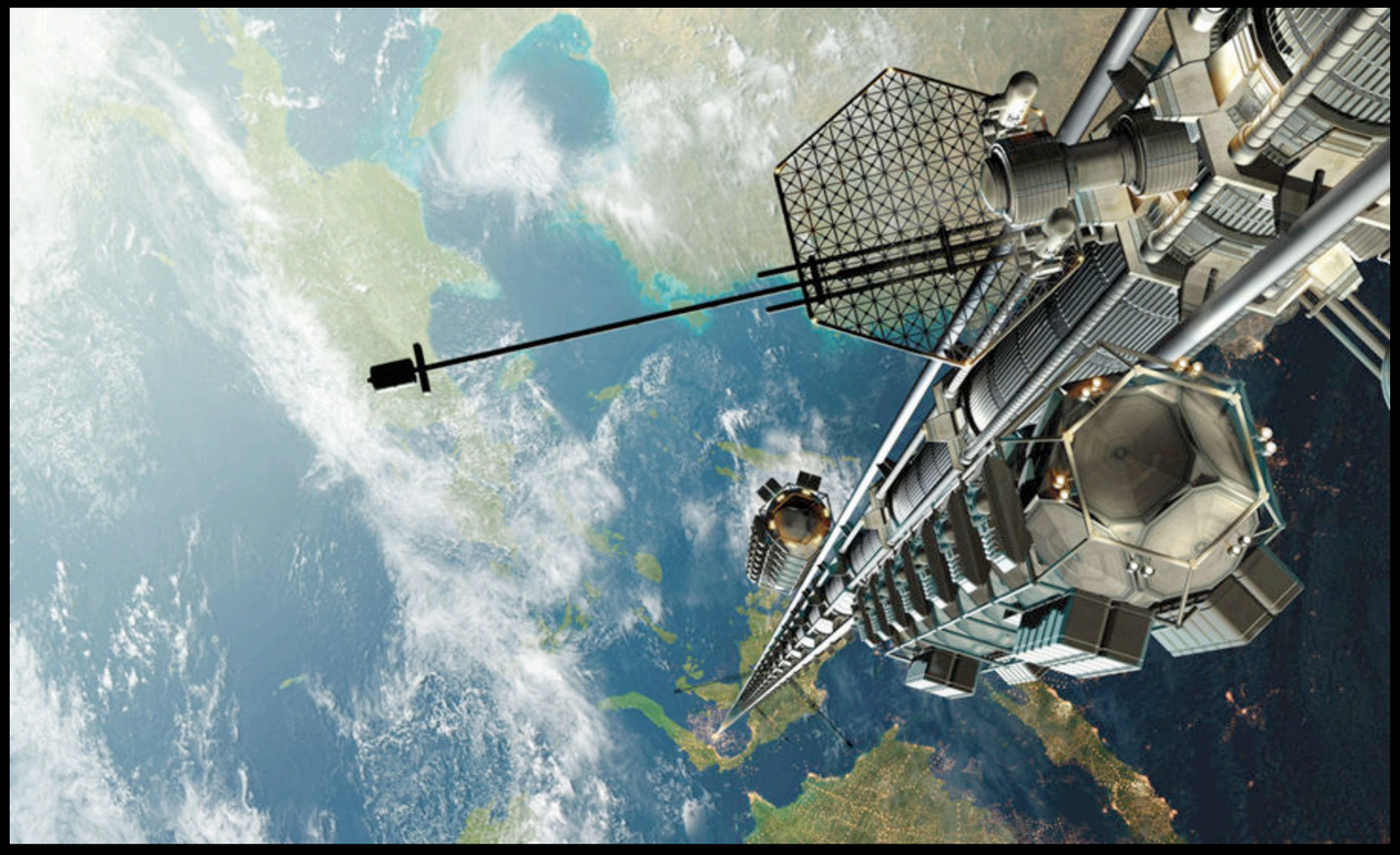
500 years ago, helicopters were a dream



Today, space elevators are.

Problem: We are in the year 2094. Many space elevators have been built. The gravitational field of the earth is $(-x,-y,-z)/(x^2 + y^2 + z^2)^{3/2}$.

What work is necessary to go from the earth ground (0,0,6) to the point (0,0,66)?



(units are 1000 km)

Instead of computing the line integral, it is better to realize that F has a potential and taking the potential difference.

The potential is $1/\sqrt{x^2+ y^2+ z^2}$

Gradient field

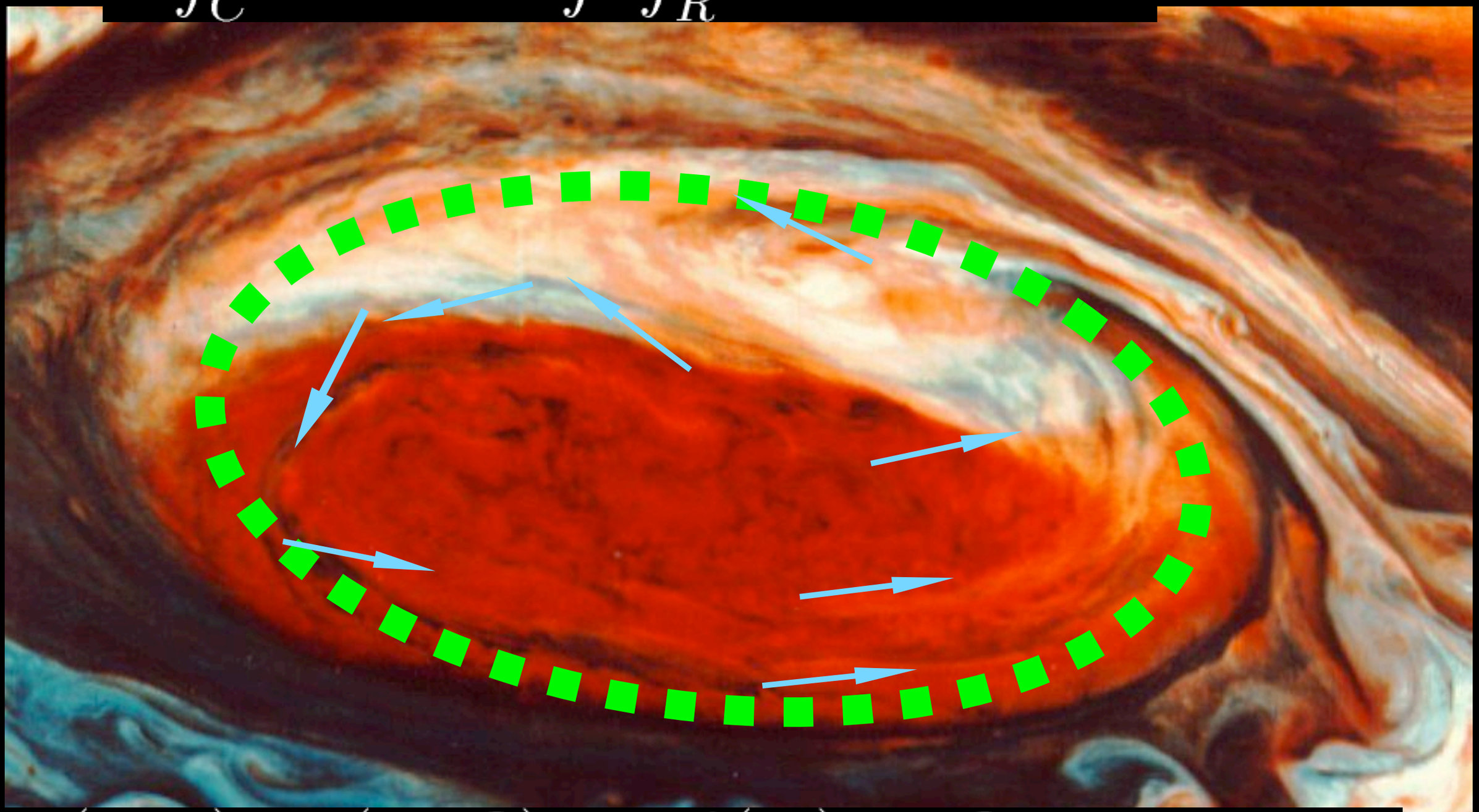
Conserv-ative

Closed Loop property

Mixed derivative test

Greens Theorem

$$\int_C F \cdot dr = \int\int_R \text{curl}(F)\ dxdy$$

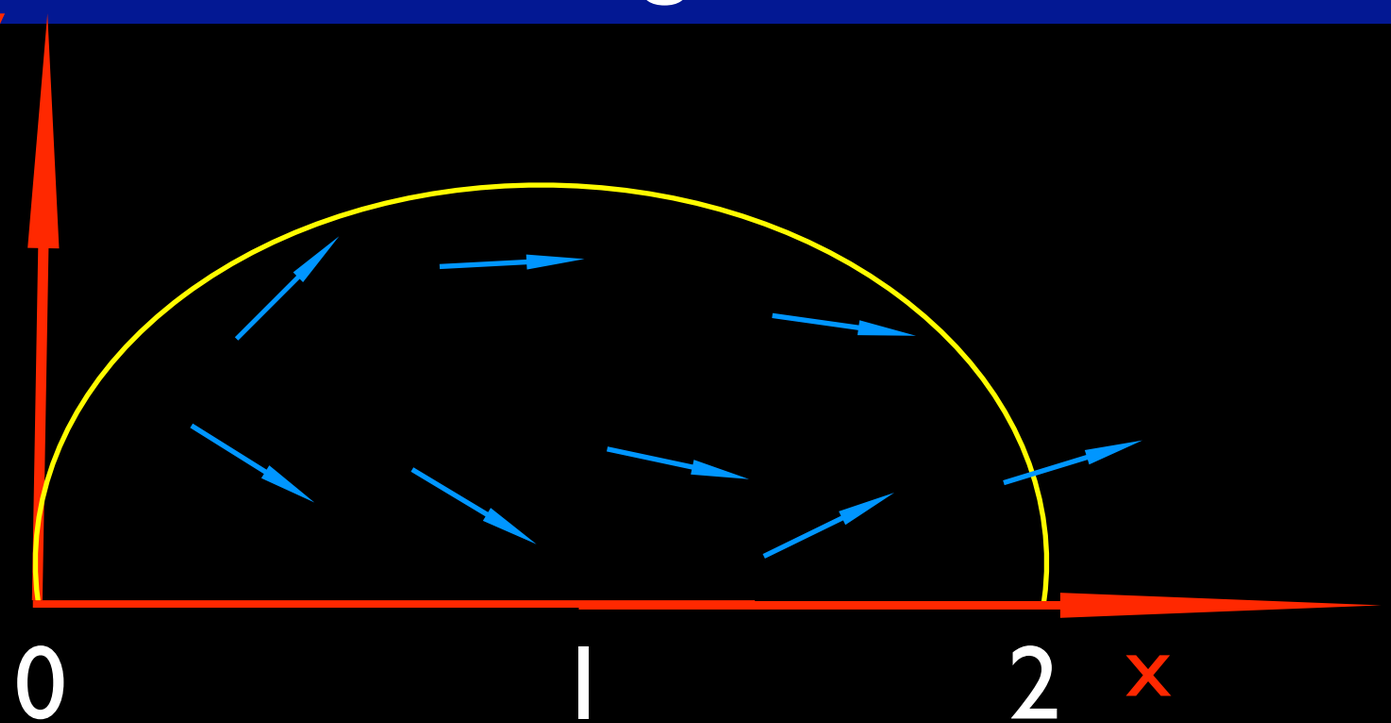


$F(x, y) = (P, Q)$, $\text{curl}(F) = Q_x - P_y$.

A line integral problem

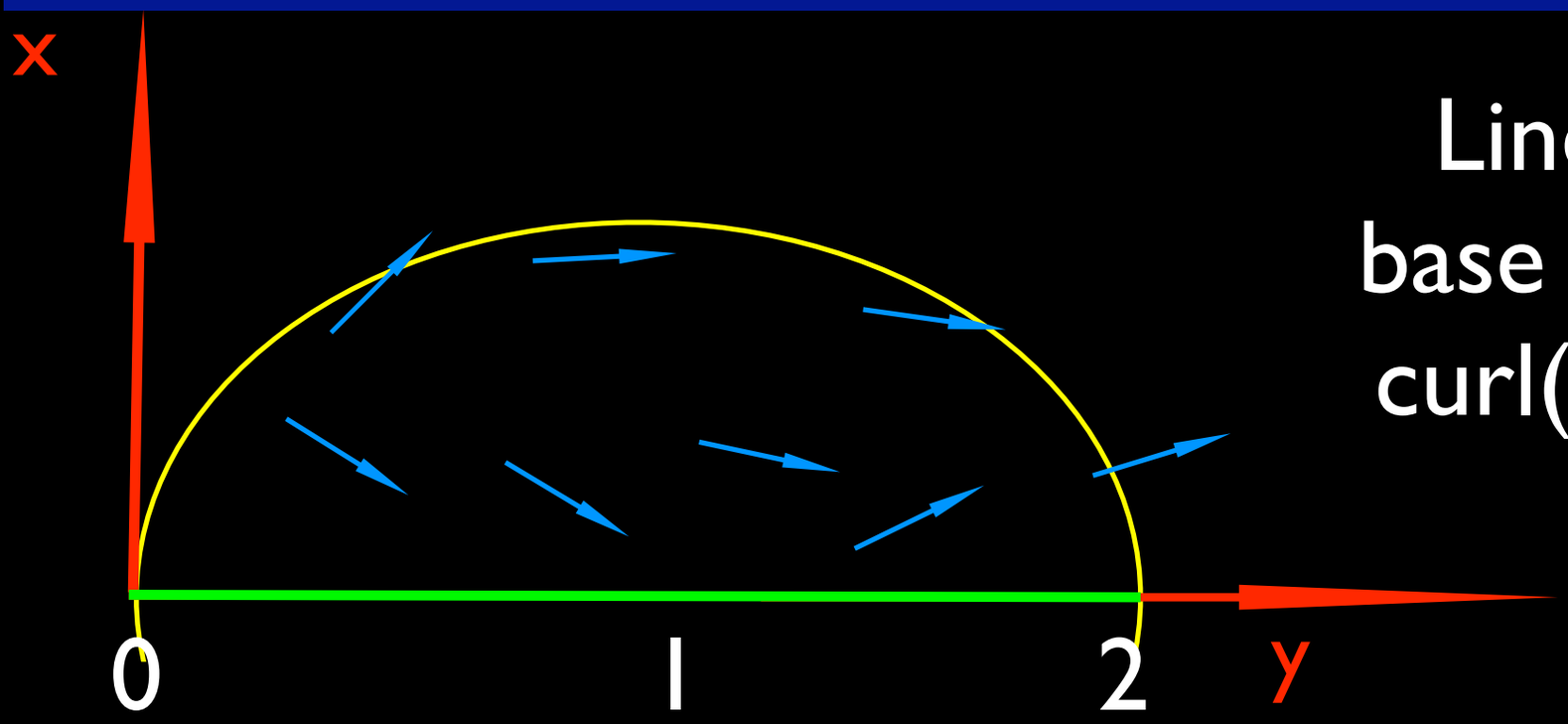
Problem

Find the line integral of the vector field
$F(x,y)=(y^2\cos(x) +3, x+2y \sin(x))$
along the upper semi circle (counterclockwise)
given below.



Use Green!

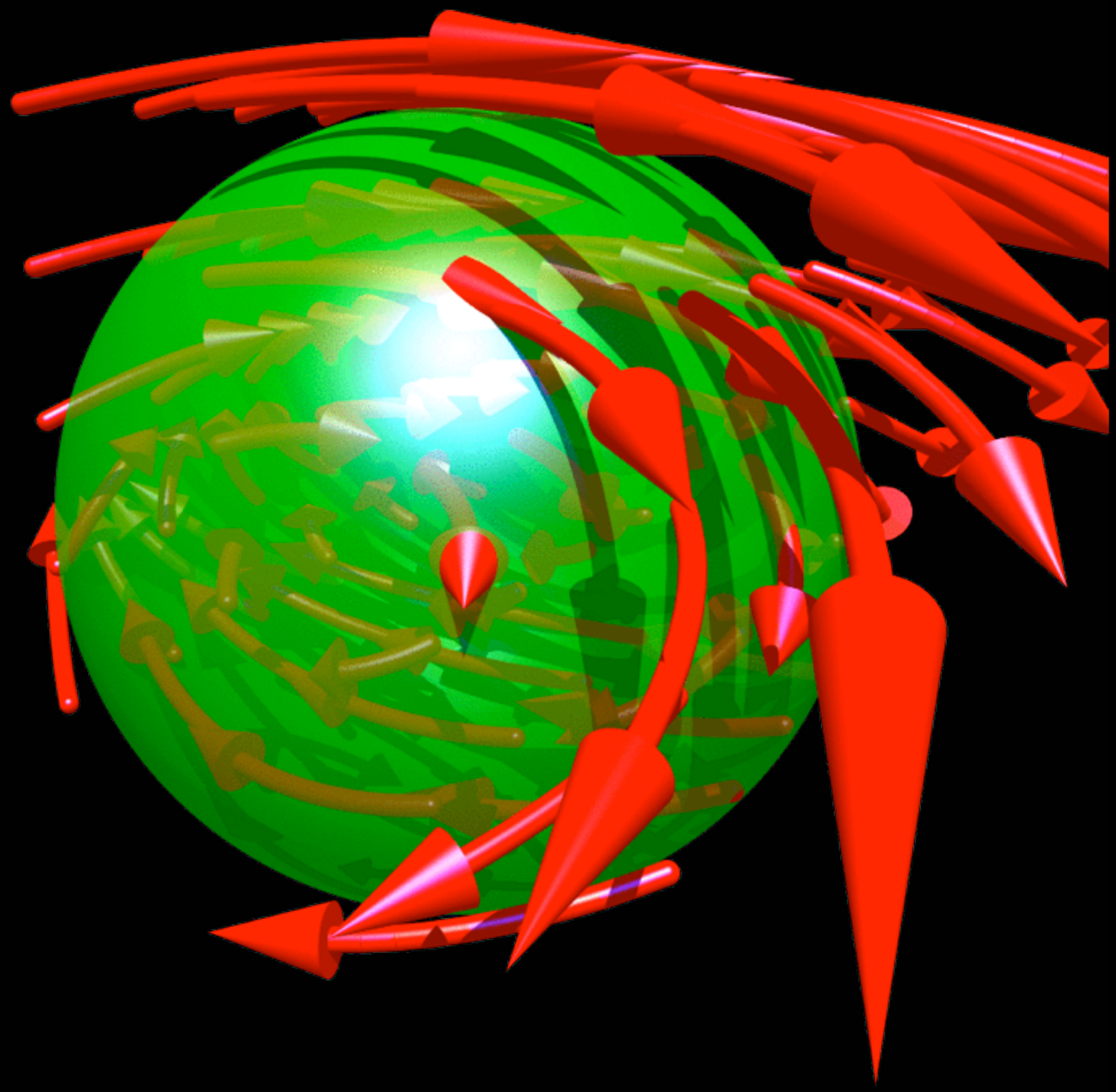
The vector field
$F(x,y)=(y^2\cos(x) +3, x+2y \sin(x))$
has curl(F)=1



Line integral along base is 6. Integral of curl(F) over region is pi/2.

Answer: X=pi/2-6

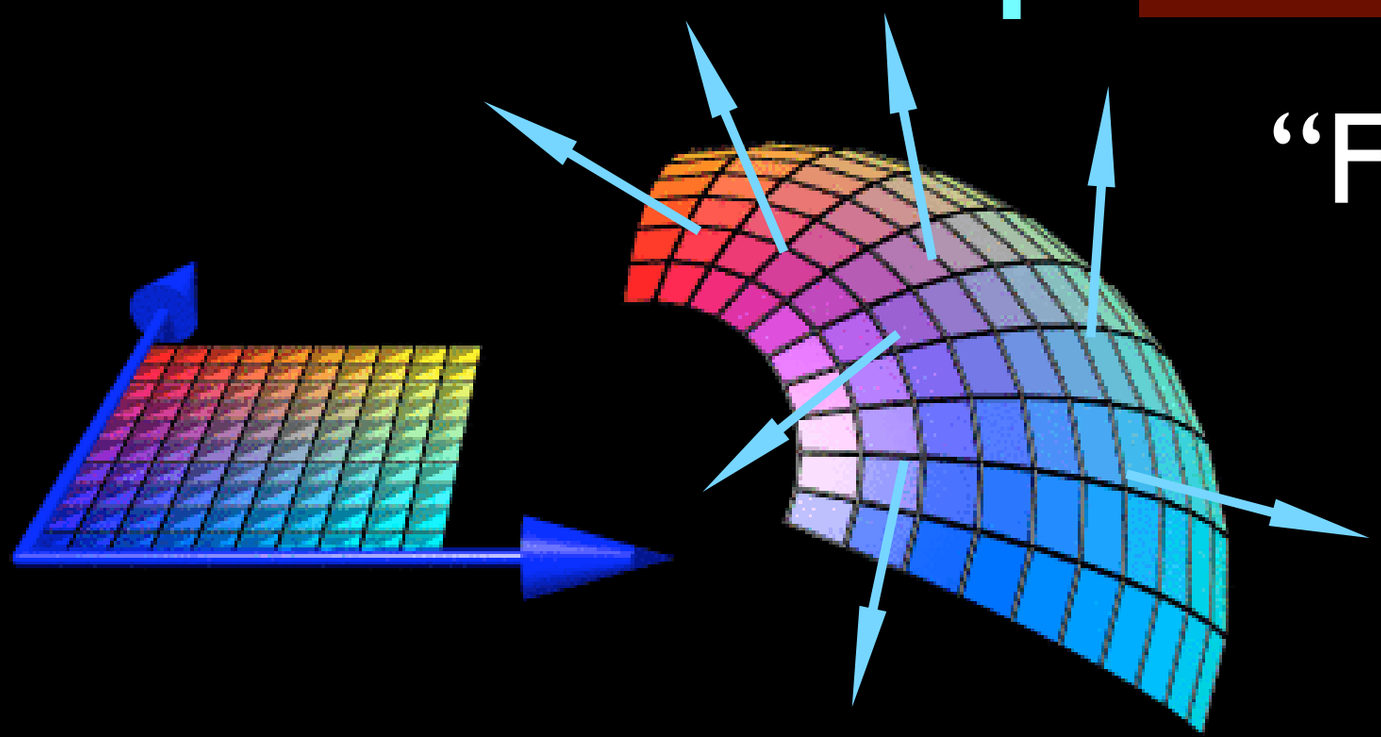
Flux



Flux integrals

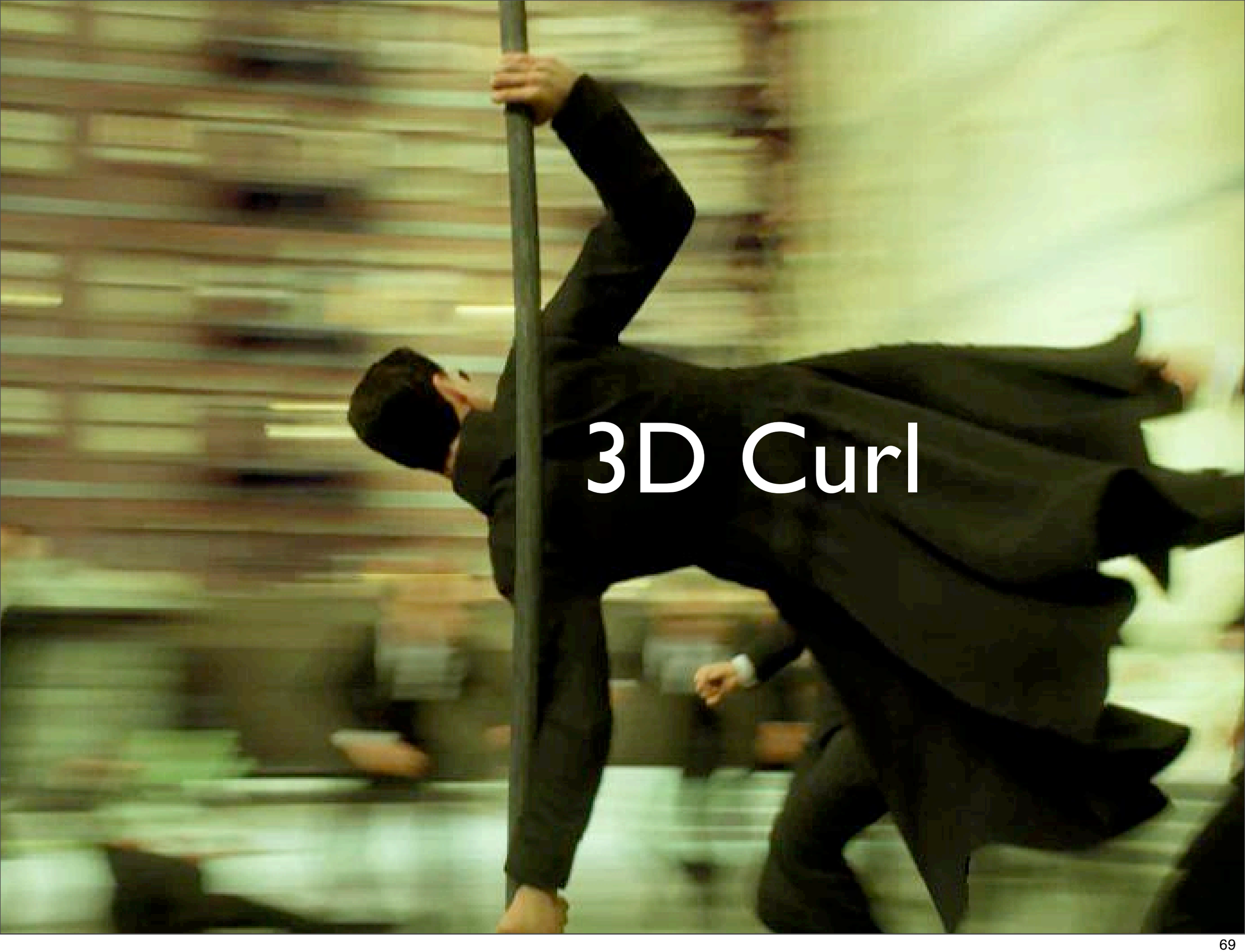
F(r(u,v)) "Fluid velocity"

$$F(r(u,v)) \cdot r_u \times r_v =$$



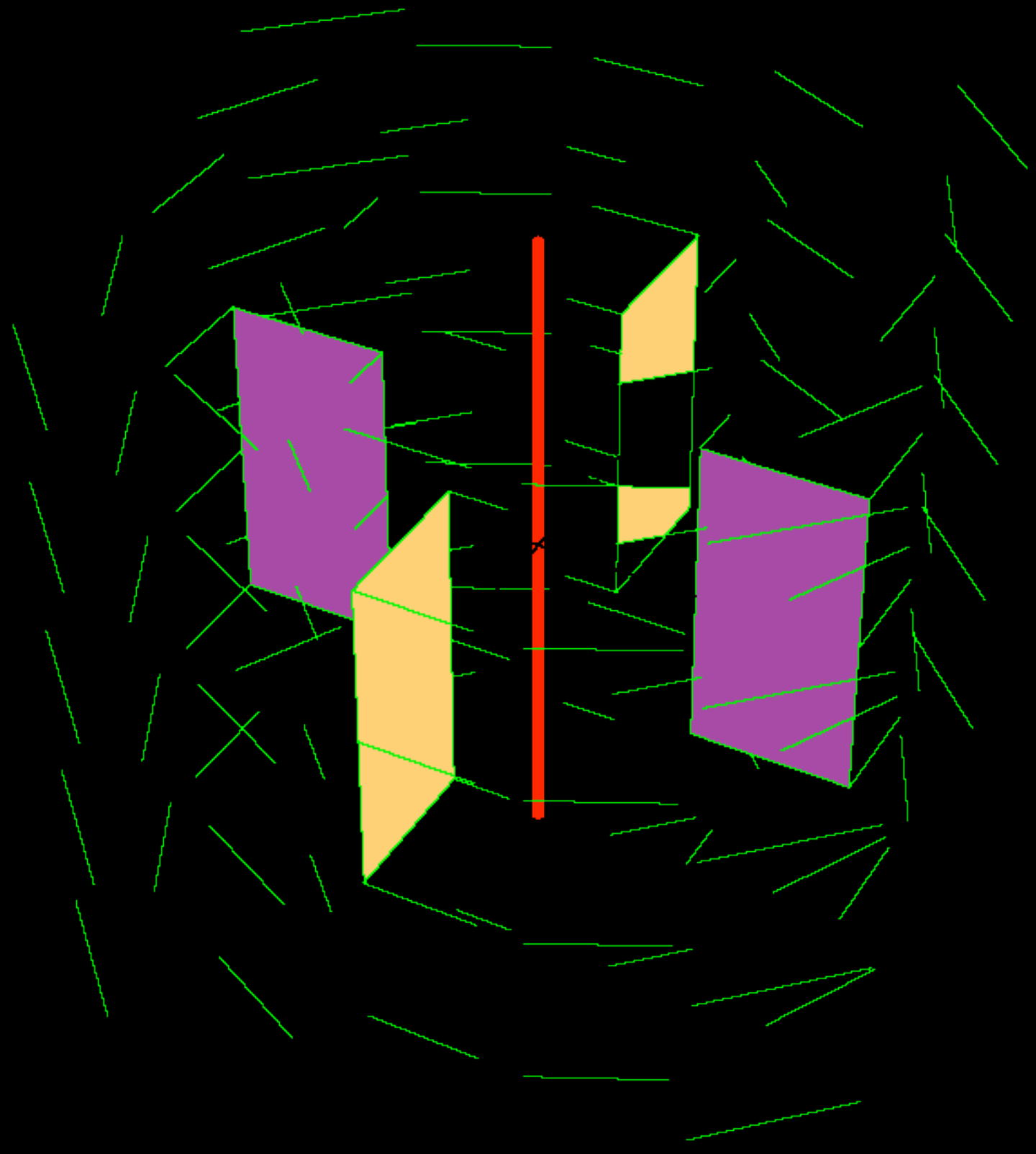
"Flux component"

Integrating this over the surface gives the flux.


3D Curl

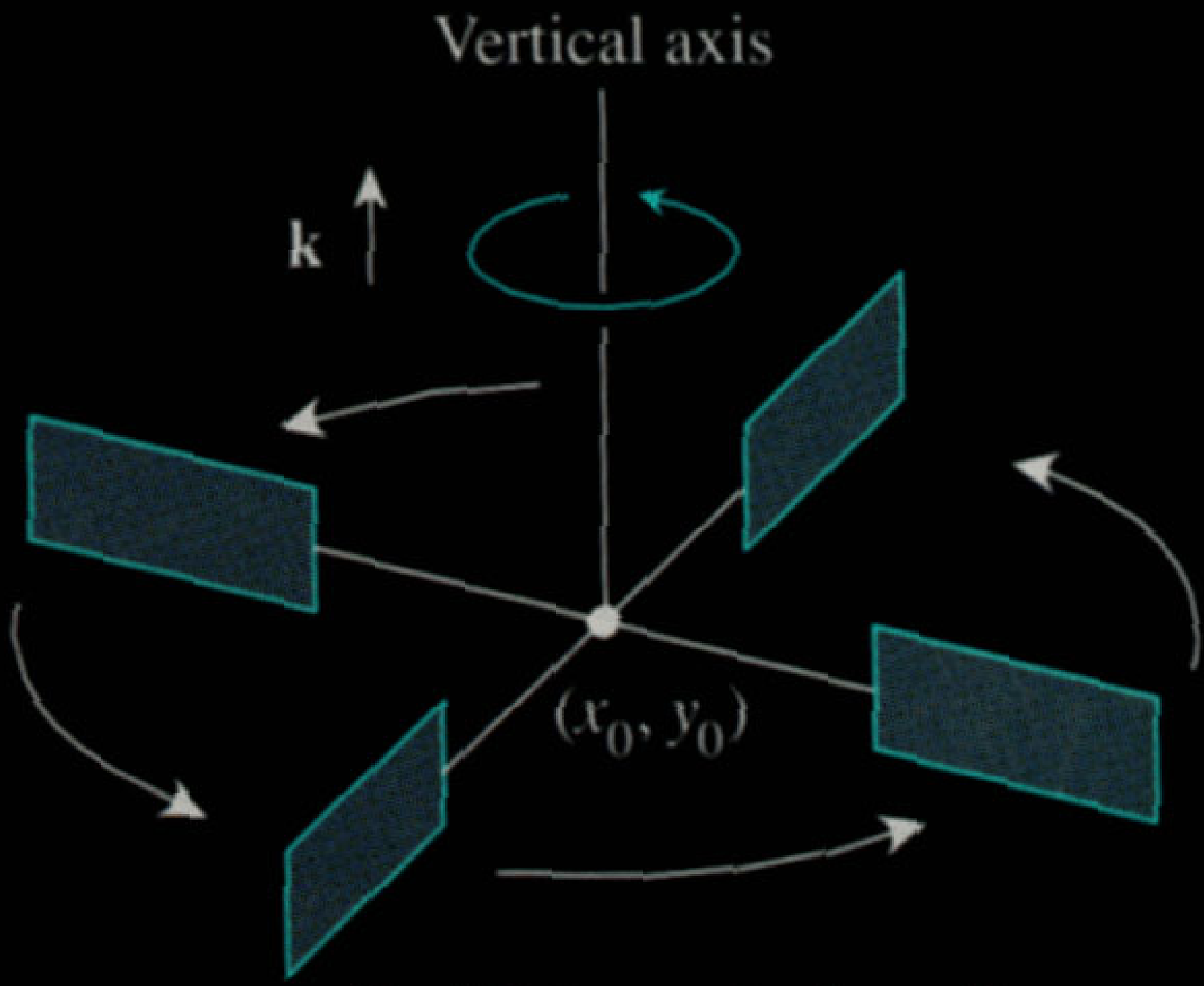
3D Curl

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix}$$



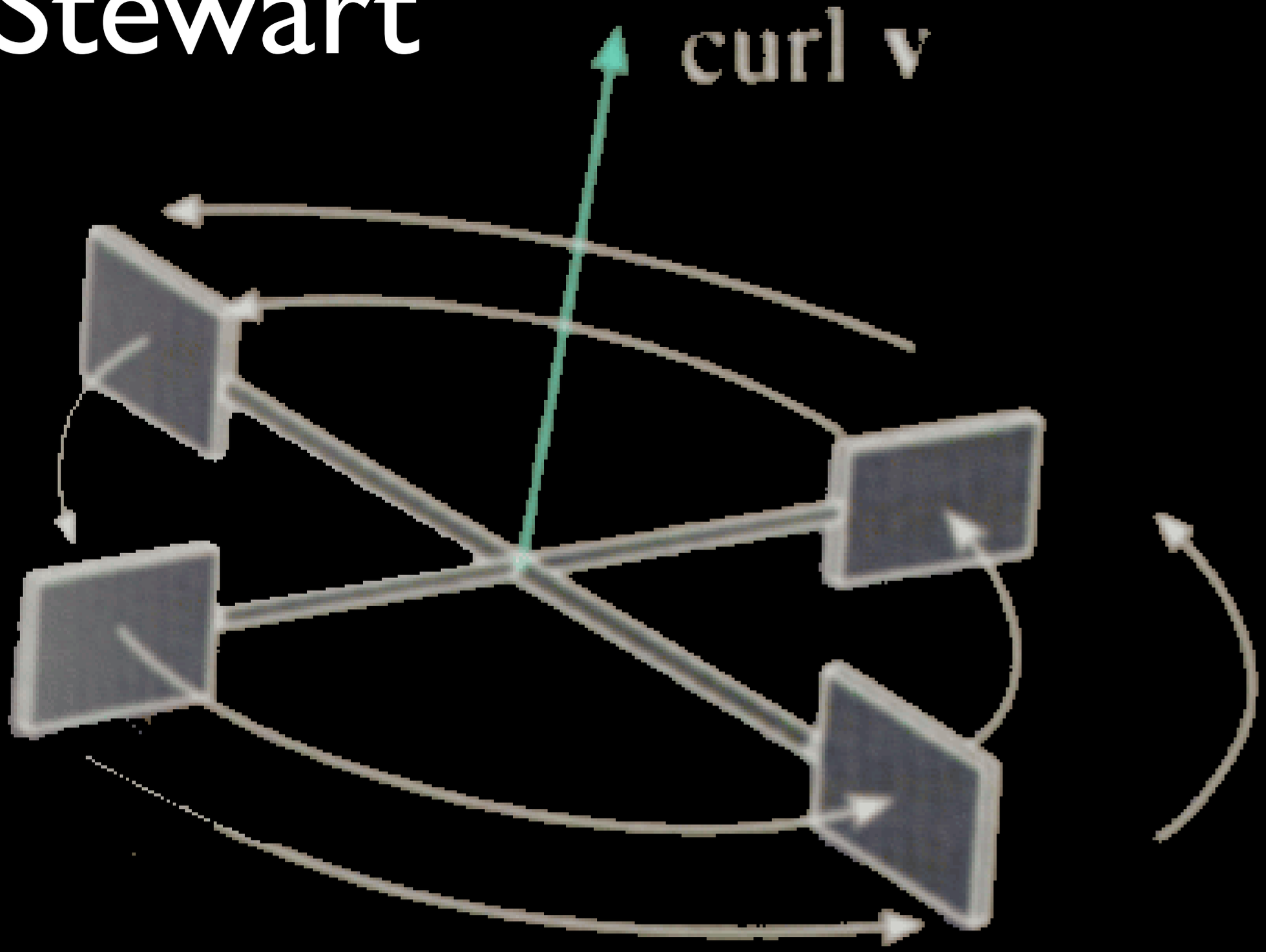
In essentially all multivariable textbooks, the paddlewheel appears.

Thomas



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation

Stewart



Edwards-Penny

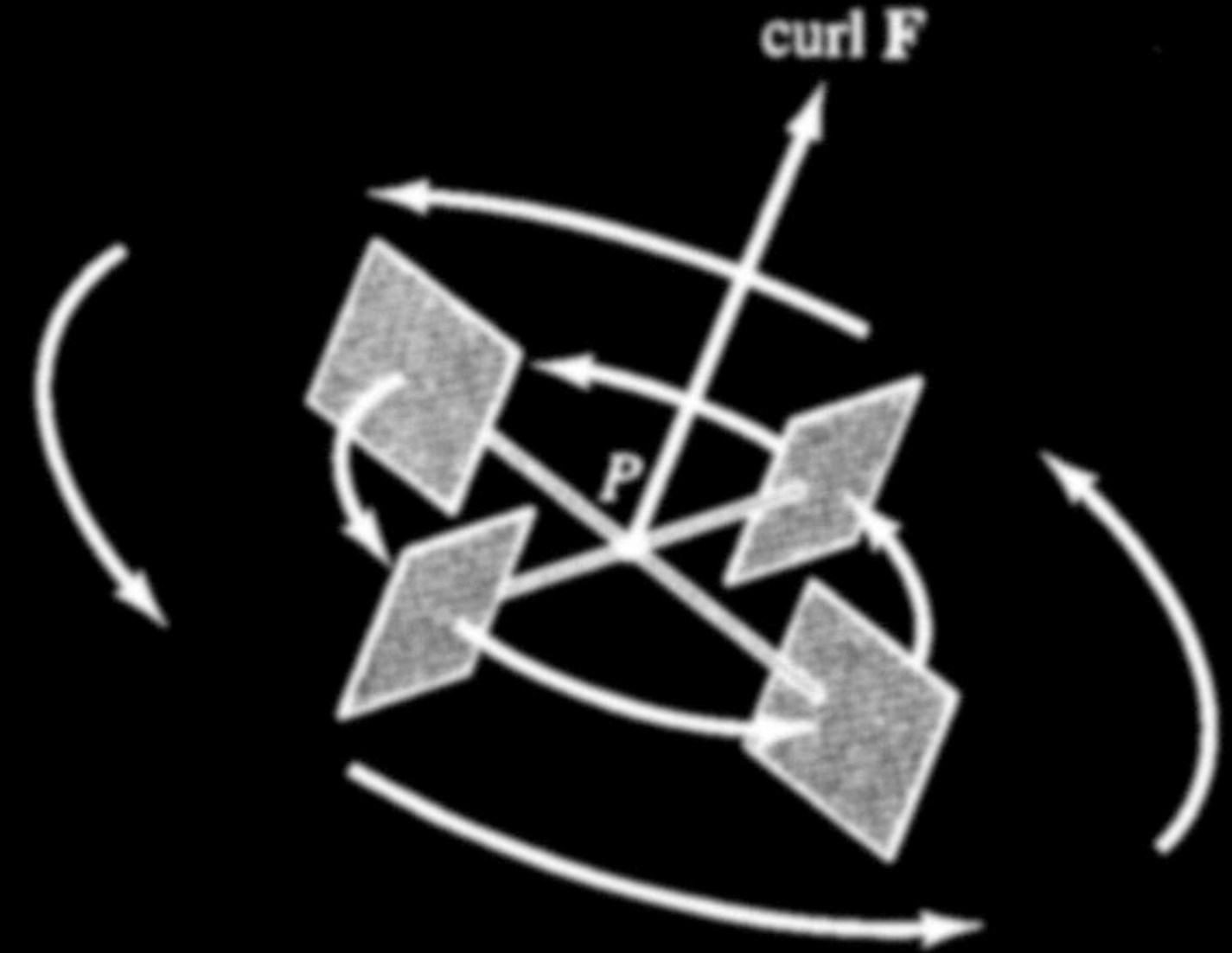


FIGURE 15.7.6 The paddle-wheel interpretation of curl **F**.

Marsden Tromba

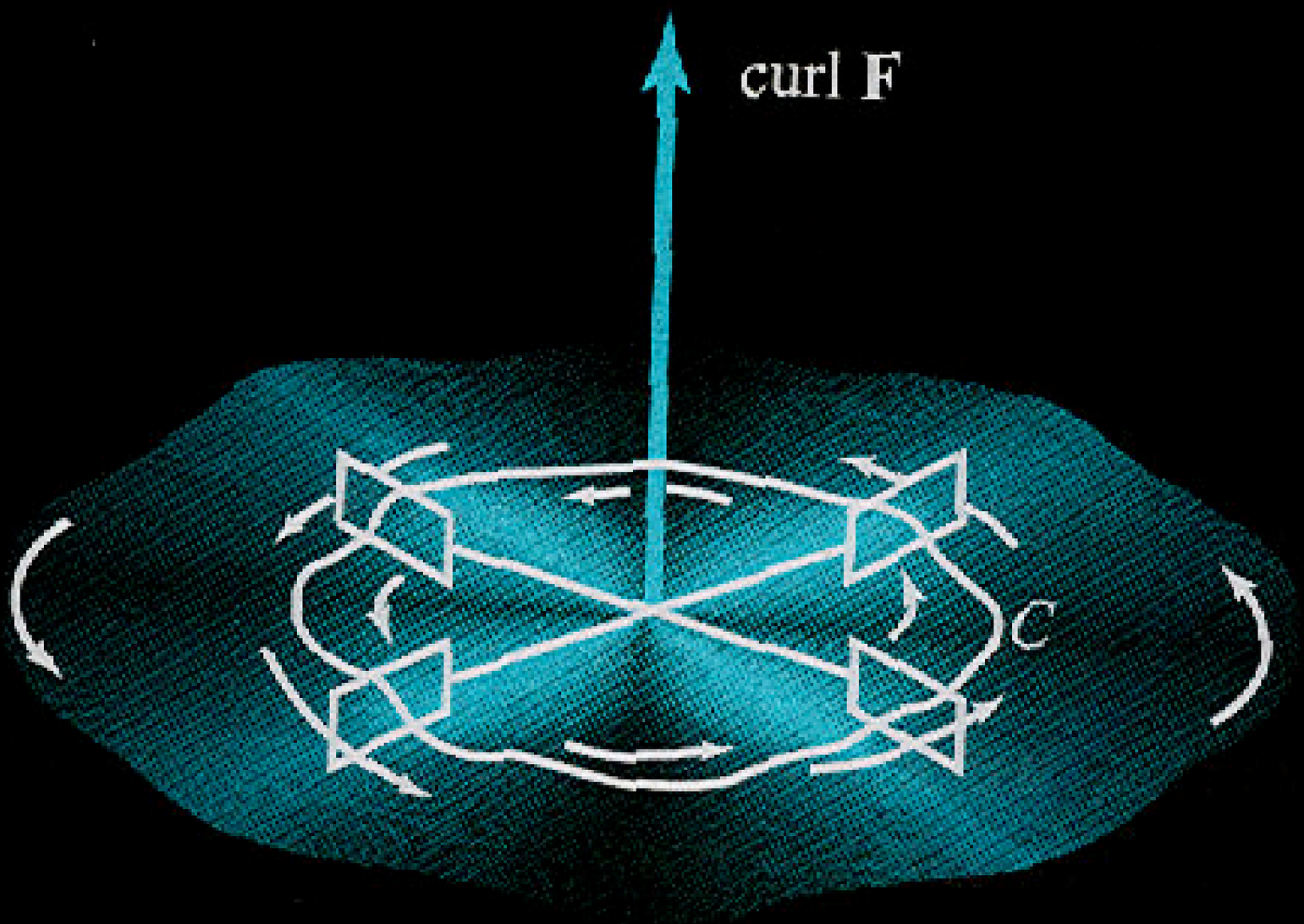
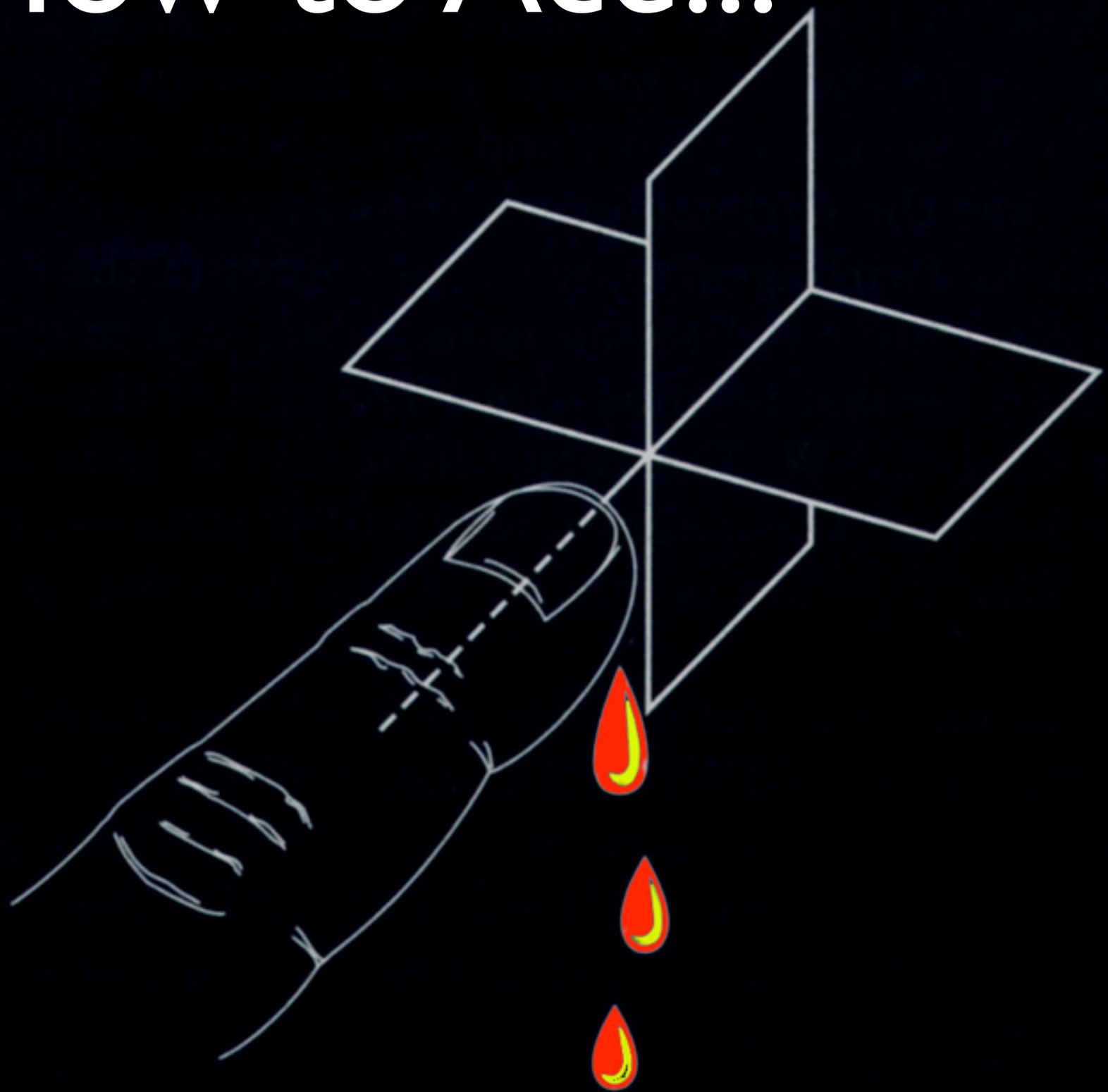


Figure 8.3.4 $\int_C \mathbf{F} \cdot$
paddle wheel in a flu
rotate around its axis

How to Ace...



Curl

The wind velocity around Harvard yard is $F(x,y,z)=(2x,\sin(x),yz)$

Is this a conservative field?

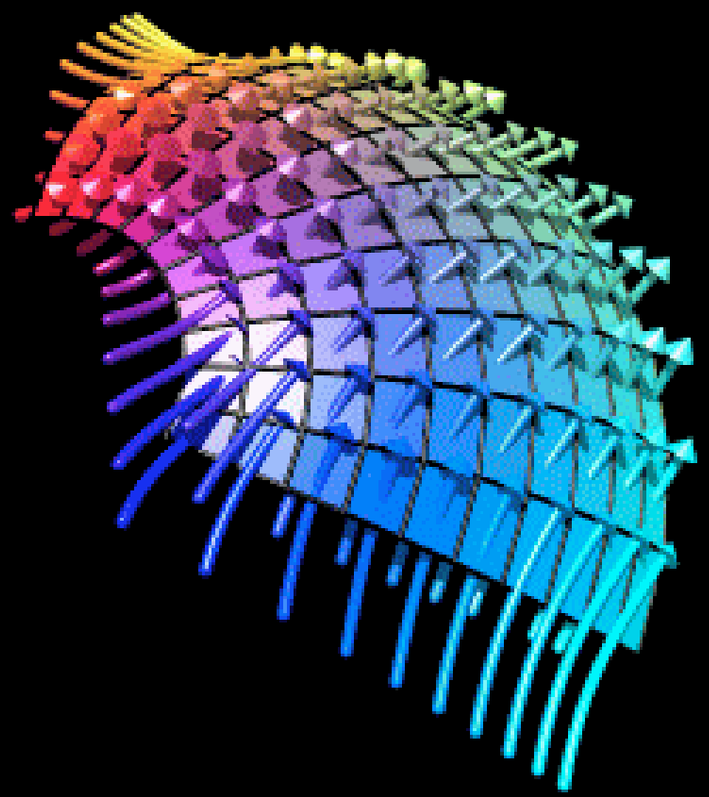
How fast rotates the wheel at the point (1,1,3)

Stokes

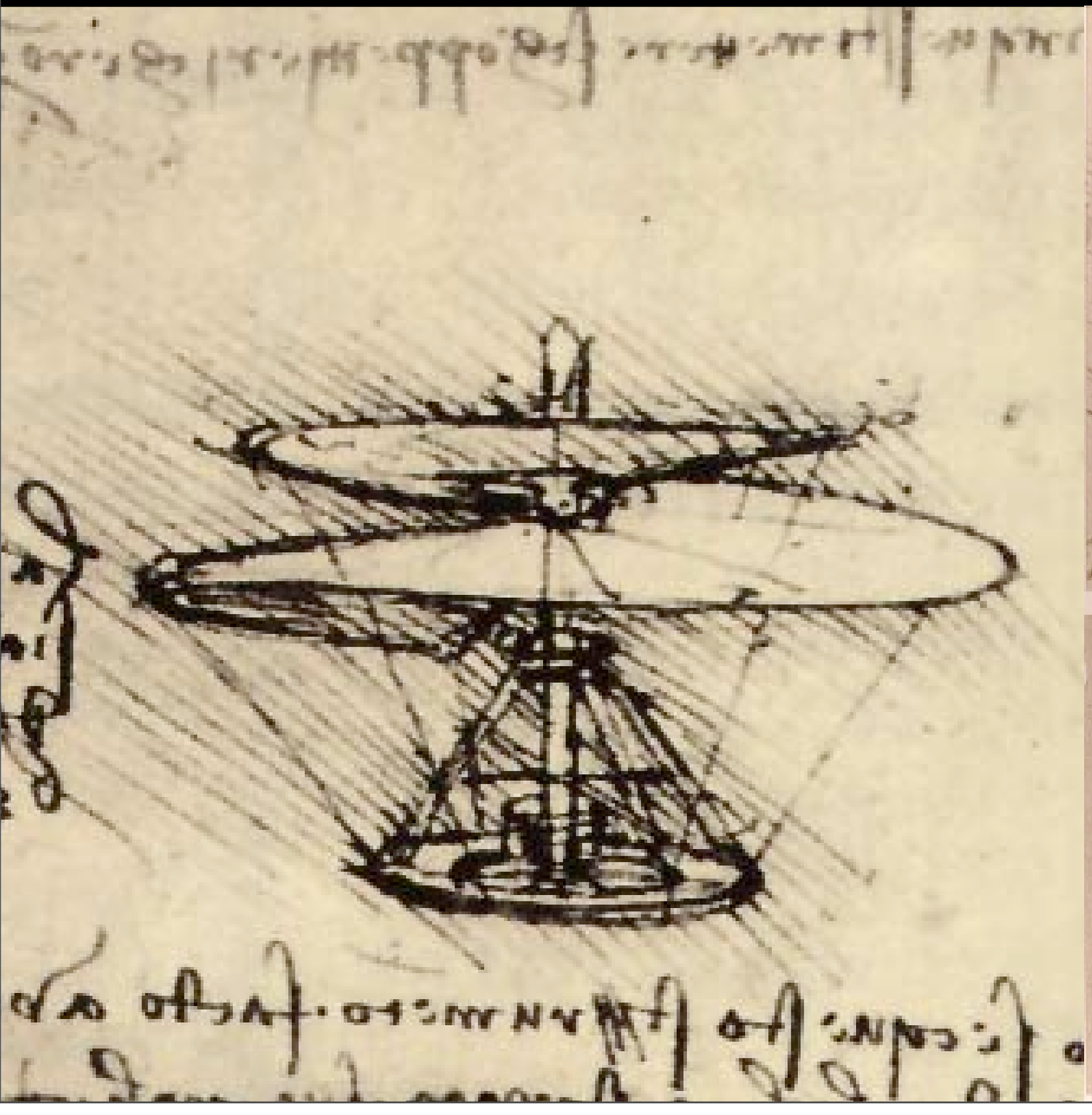
Stokes Theorem

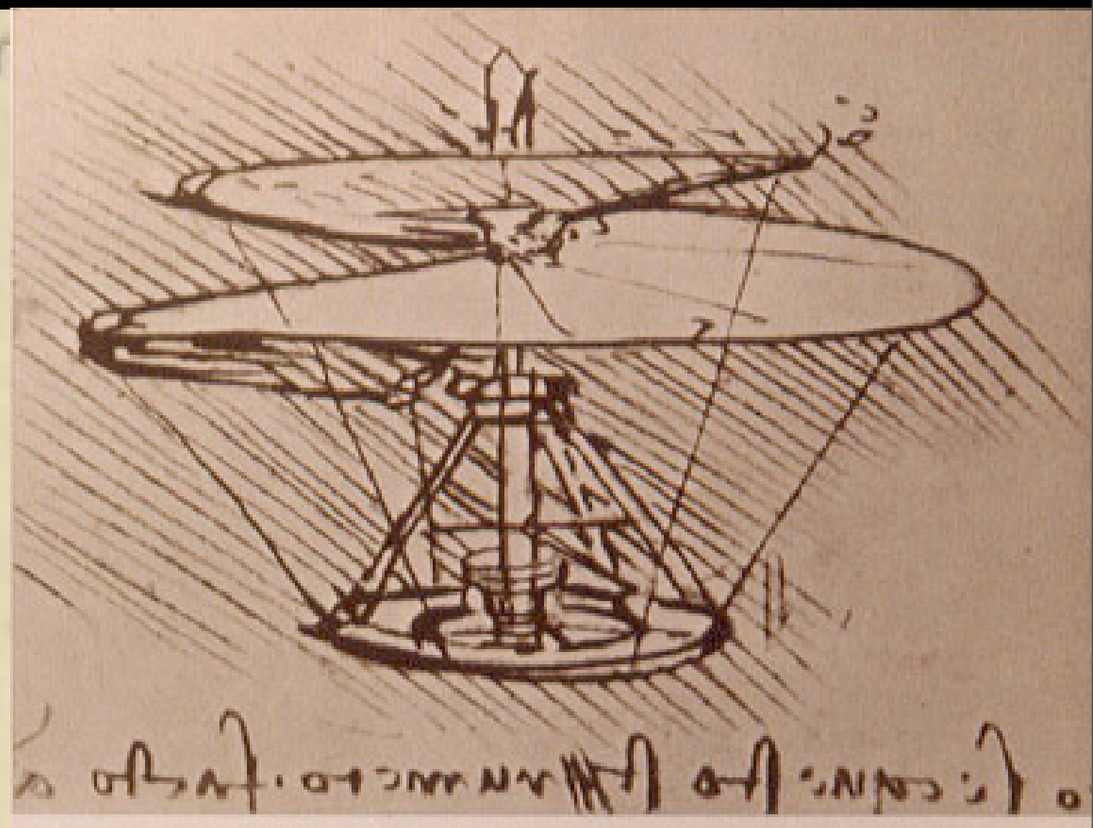
$$\mathrm{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

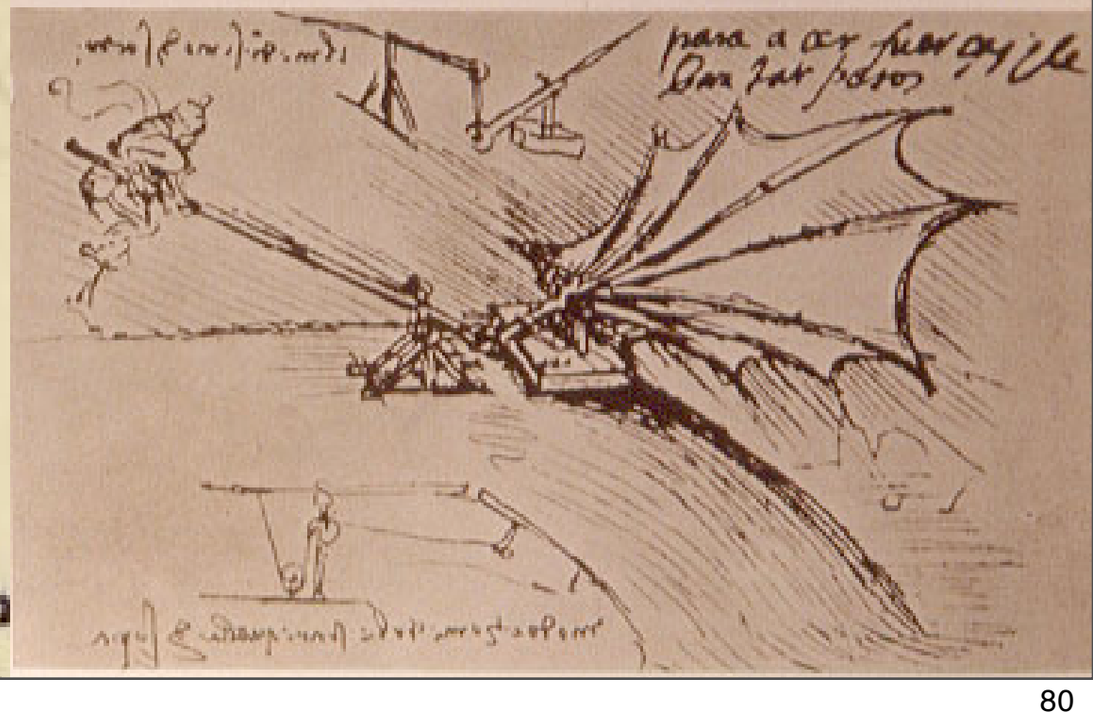
$$\int_C F \cdot dr = \int \int_S \mathrm{curl}(F) \cdot dS$$



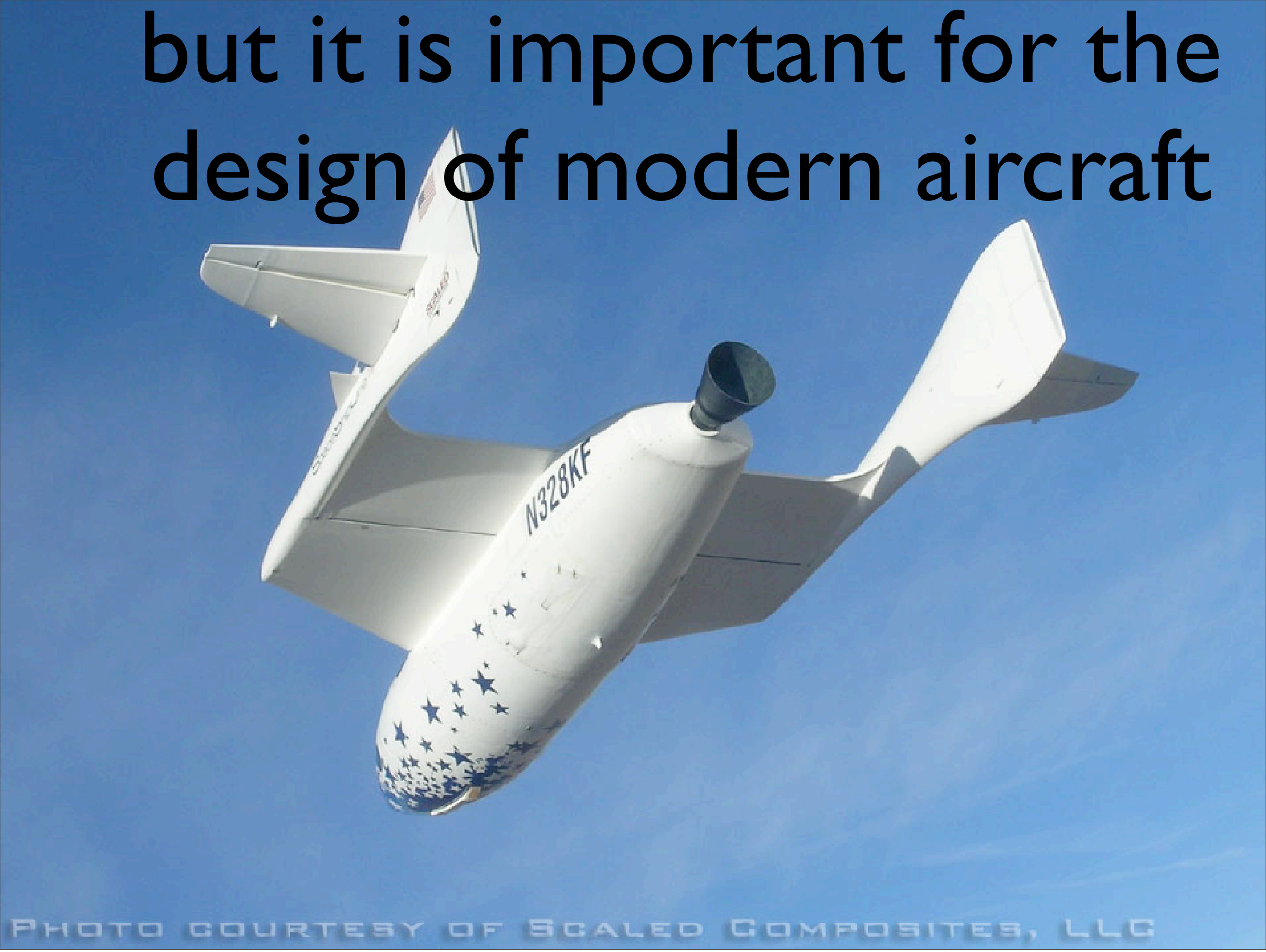
This result was not known to Da Vinci..





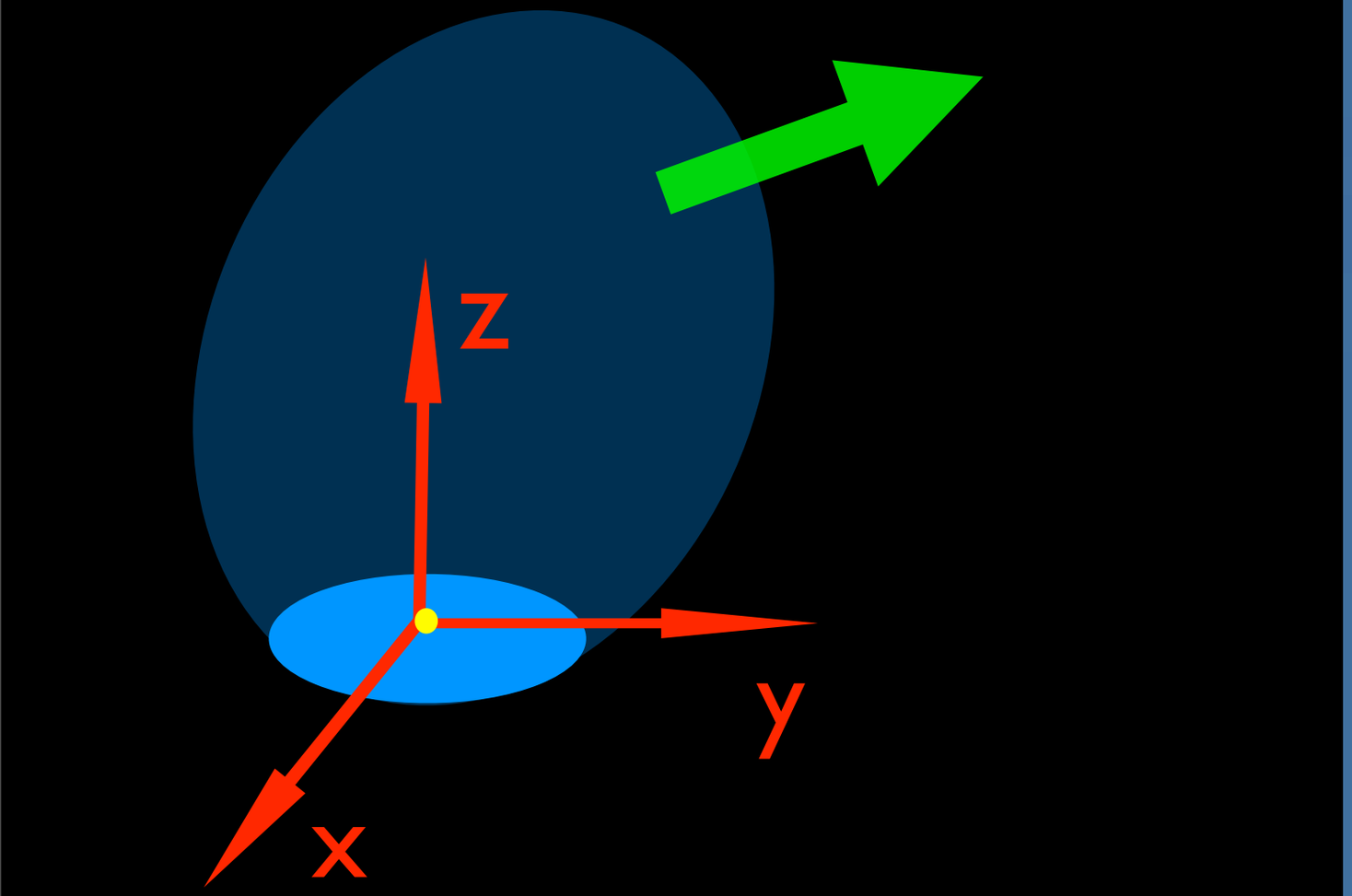


but it is important for the design of modern aircraft



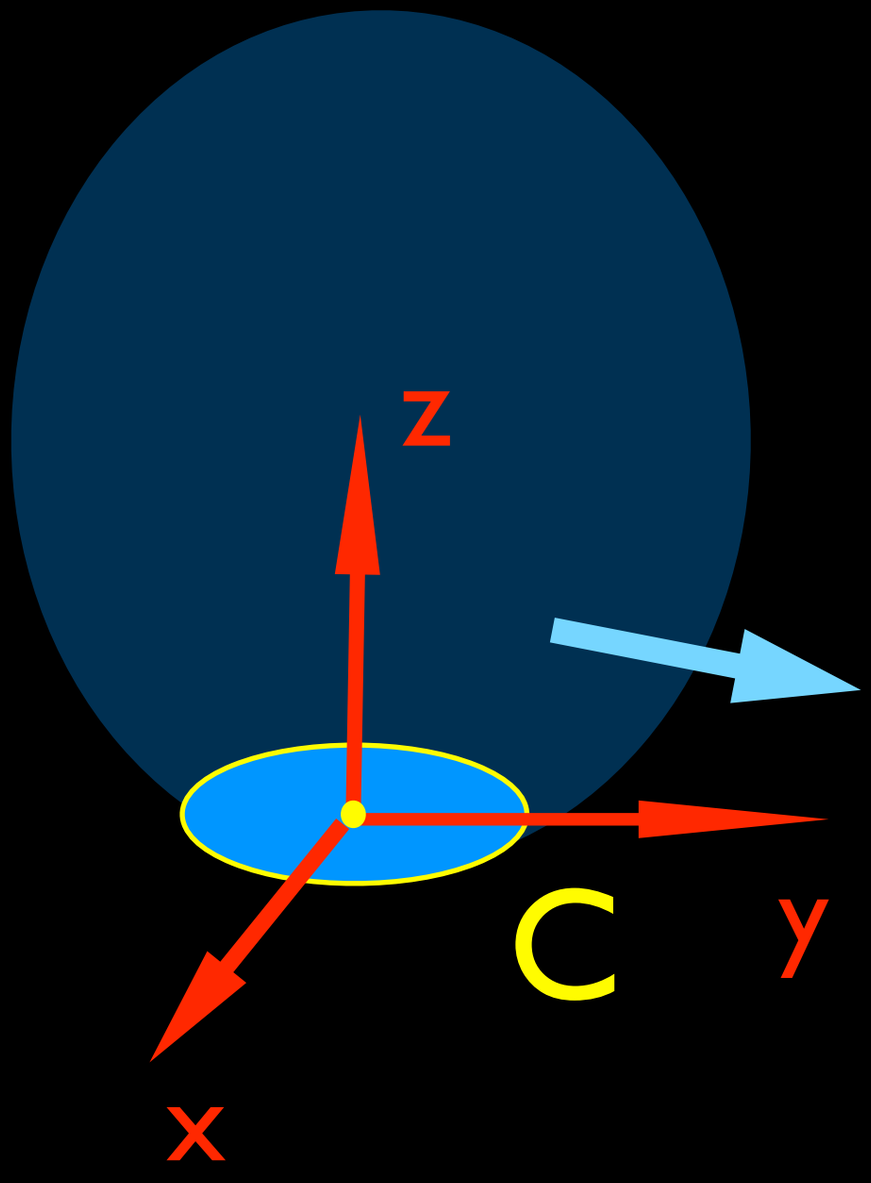
Blimp Problem

The velocity field in a hot air balloon is F=curl(G), where G=(-y,x,0). What is the flux of F through the surface of a hot air balloon for which the opening is the unit circle in the xy plane



Use Stokes Theorem

The flux of curl(F) through the balloon is equal to the lineintegral of F=(-y,x,0) along the boundary curve C.



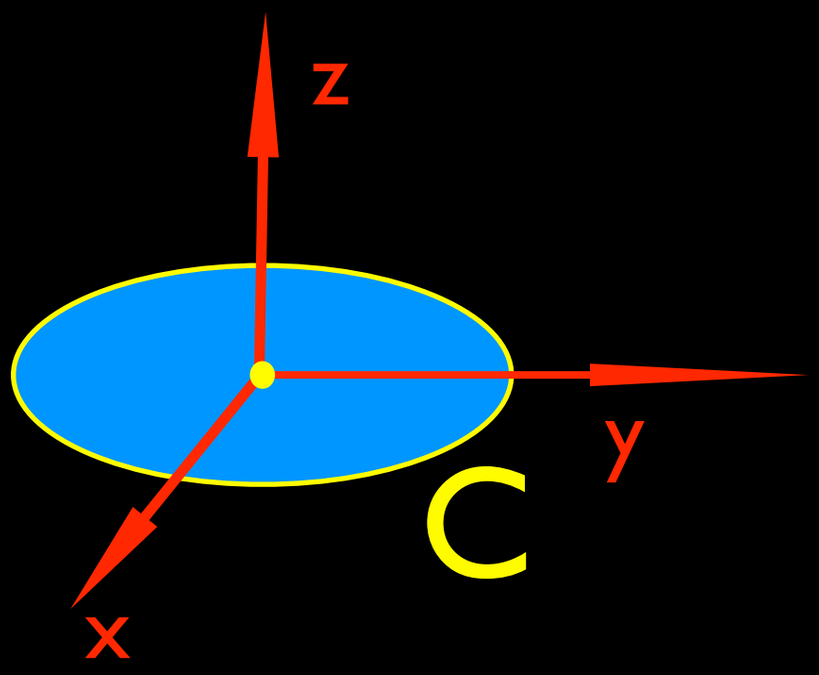
Use Stokes Theorem

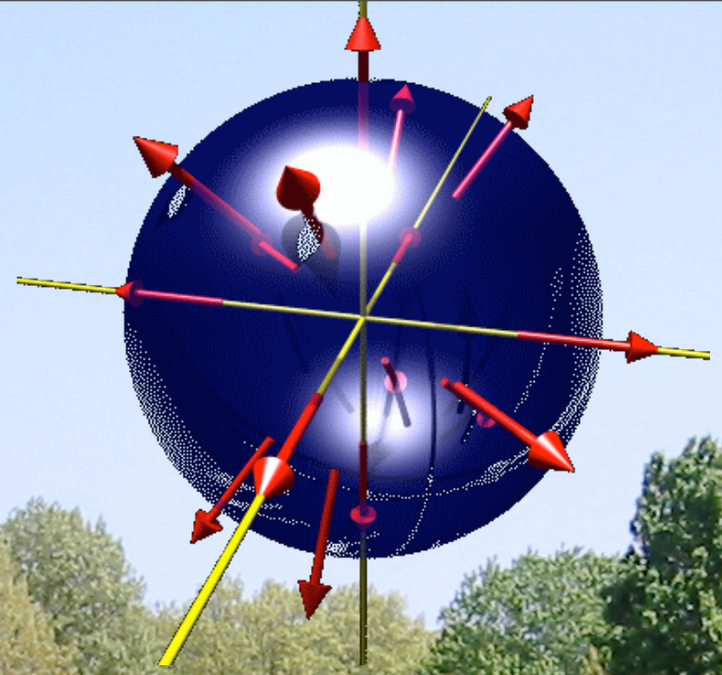
C: r(t) = (cos(t),sin(t),0). r'(t)=(-sin(t),cos(t),0)

F(r(t))=(-sin(t),cos(t),0)

F(r(t)) . r'(t)= 1

The line integral is 2pi, and so is the flux.





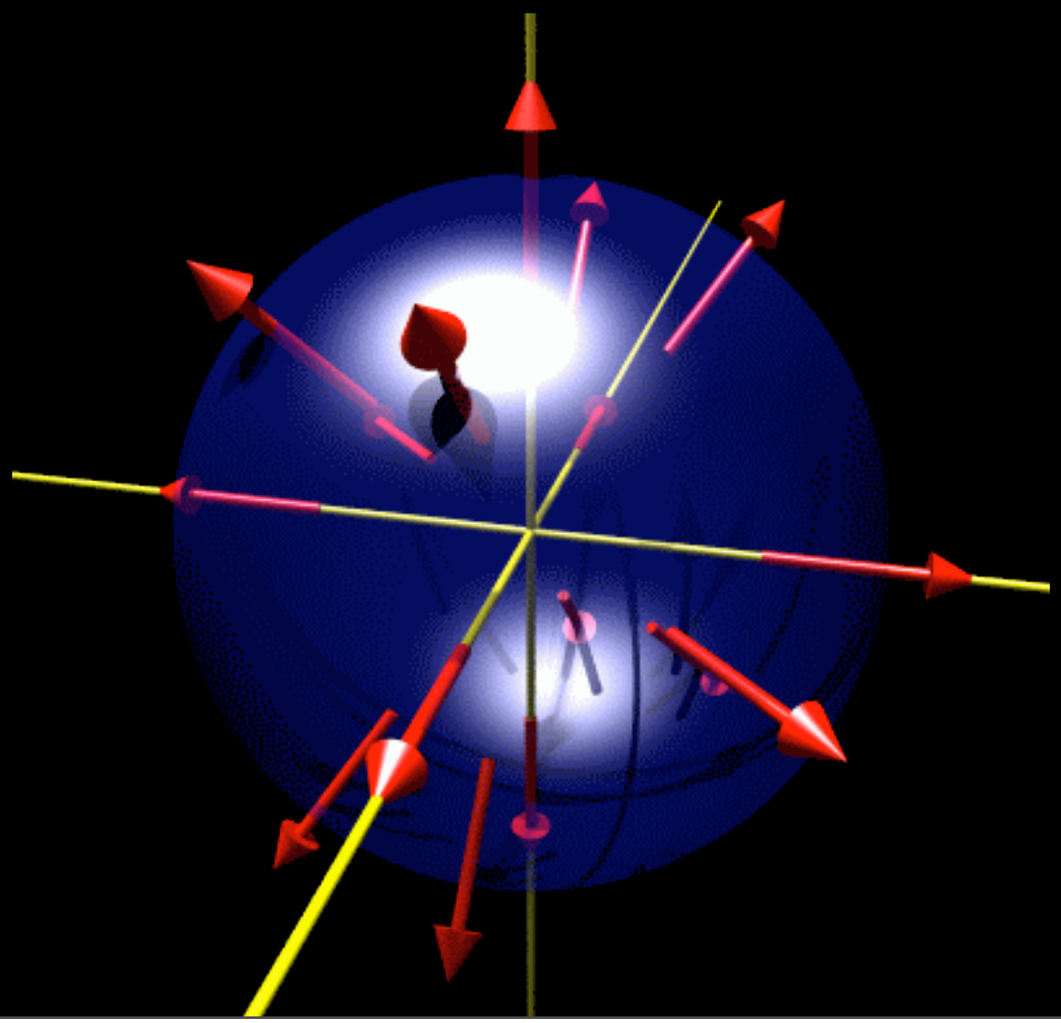
The divergence theorem

Divergence Theorem

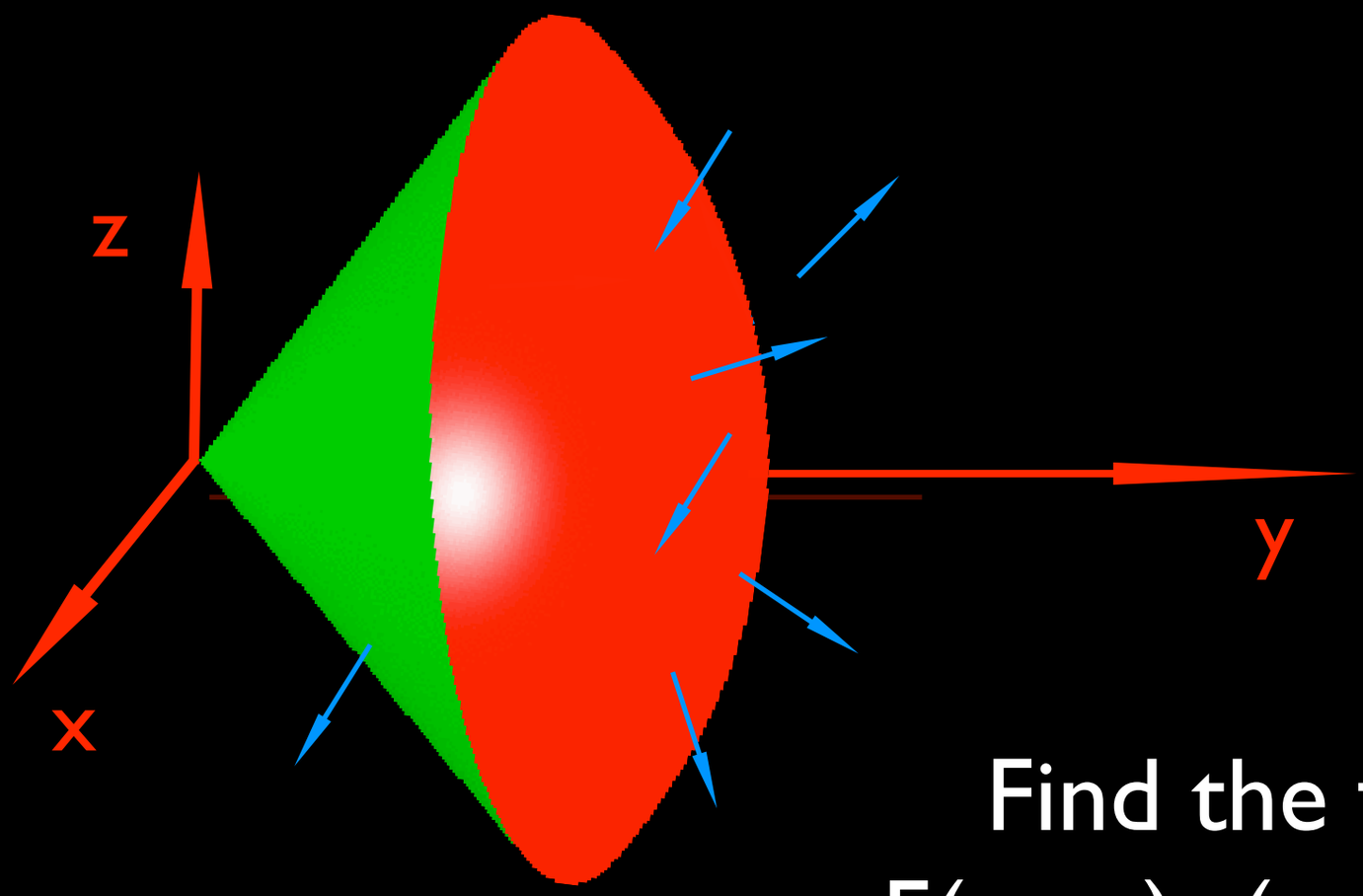
Divergence Theorem

$$\operatorname{div}(F) = P_x + Q_y + R_z$$

$$\int\int_S F \cdot dS = \int\int\int_E \operatorname{div}(F)\ dV$$



Problem



Find the flux of the vector field
F(x,y,z)=(x-sin(y),y+1,log(3+y sin(x))
through the cone |x|=|z| intersected
with
y>0 and 0<|(x,y,z)|<2

Derivatives Overview

$$\text{grad}(f) = (f_x, f_y)$$

$$\text{curl}(F) = Q_x - P_y$$

$$\text{grad}(f) = (f_x, f_y, f_z)$$

$$\text{curl}(F) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

$$\text{div}(F) = P_x + Q_y + R_z$$

Integrals Overview

line and flux integrals:

$$\int_C F \cdot dr = \int_C F(r(t)) \cdot r'(t)\, dt$$

$$\int\int_S F \cdot dS = \int\int_R F(r(u,v)) \cdot (r_u \times r_v)\, dudv$$

length and area:

$$\int_C 1 ds = \int_a^b |r'(t)|\, dt$$

$$\int\int_S 1 dS = \int\int_R |r_u \times r_v|\, dudv$$

double and triple integrals:

Integral Theorems

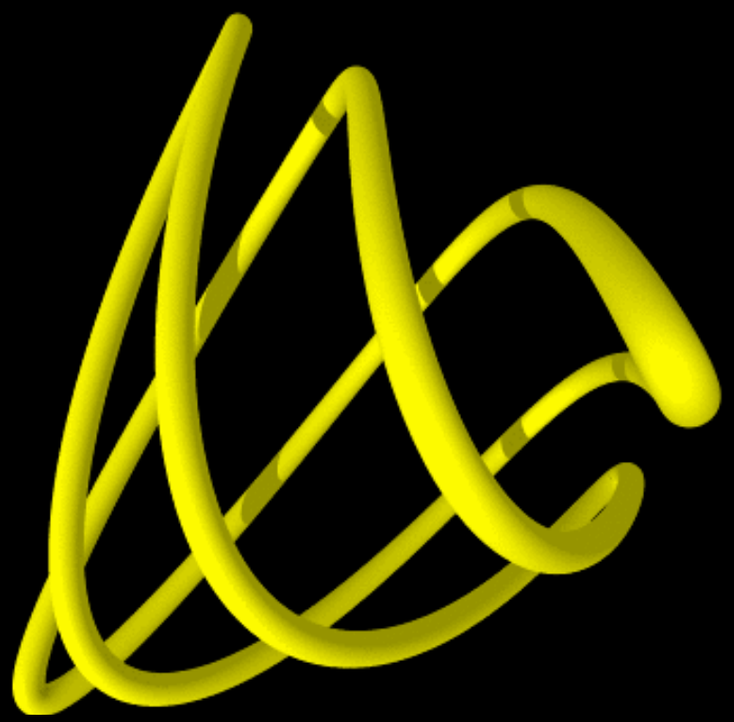
$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$\int\int_R \text{curl}(F)\ dA = \int_C F \cdot dr$$

$$\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$$

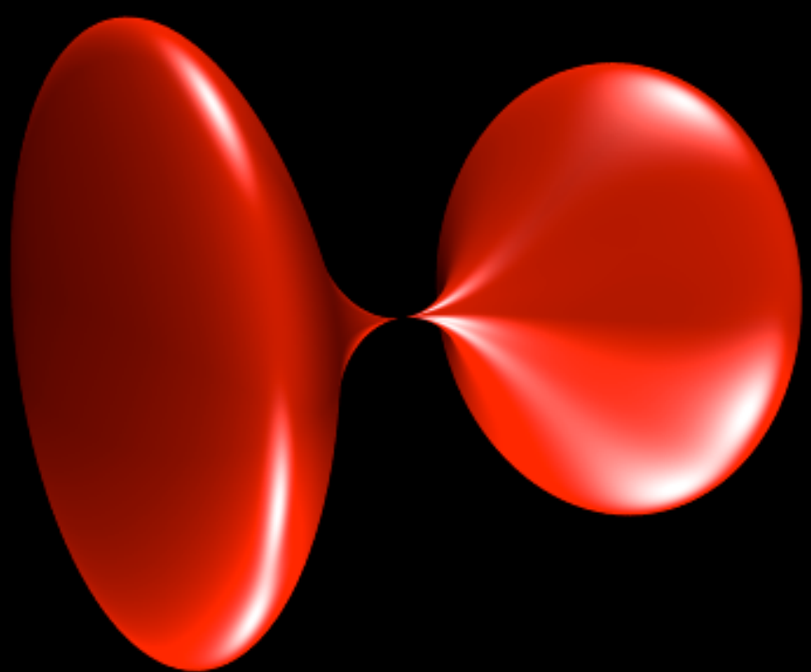
$$\int\int\int_B \text{div}(F)\ dV = \int\int_S F \cdot dS$$

Identities



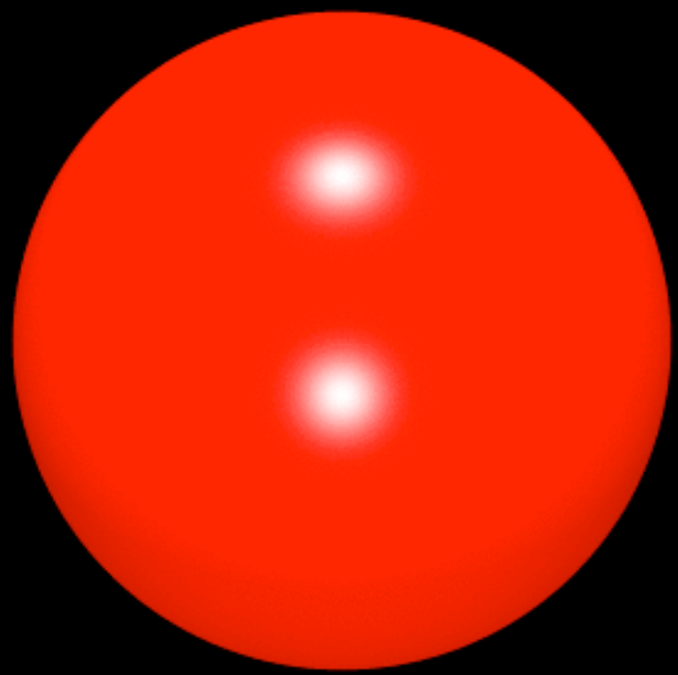
- div(curl(F))=0
- curl(grad(f))=0

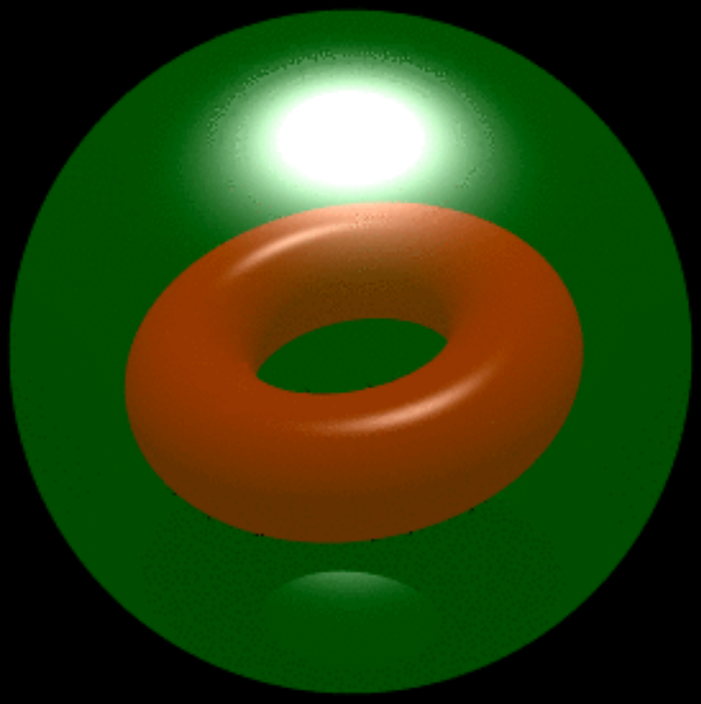
The line integral of F=grad(f) along a closed curve is zero.
(Either Stokes of fundamental Theorem of Line integrals)

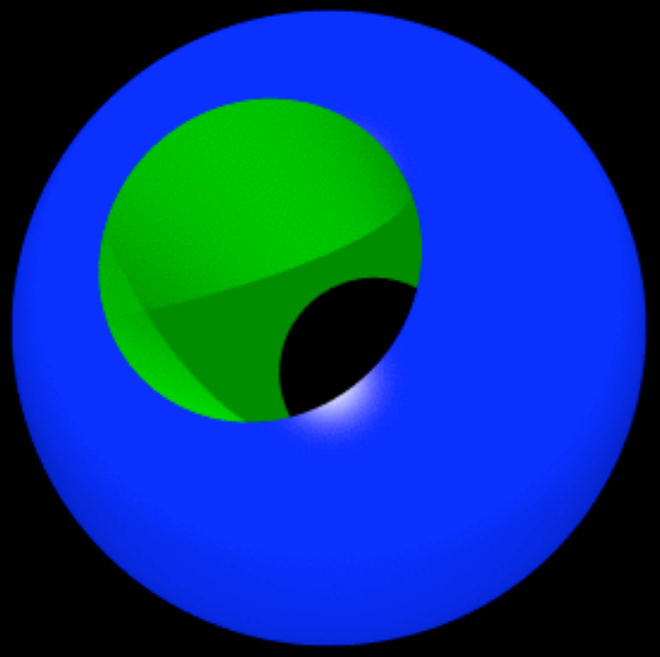


The flux of F=curl(G) through a closed surface is zero.
(Either Divergence theorem of Stokes Theorem)

Simply connected

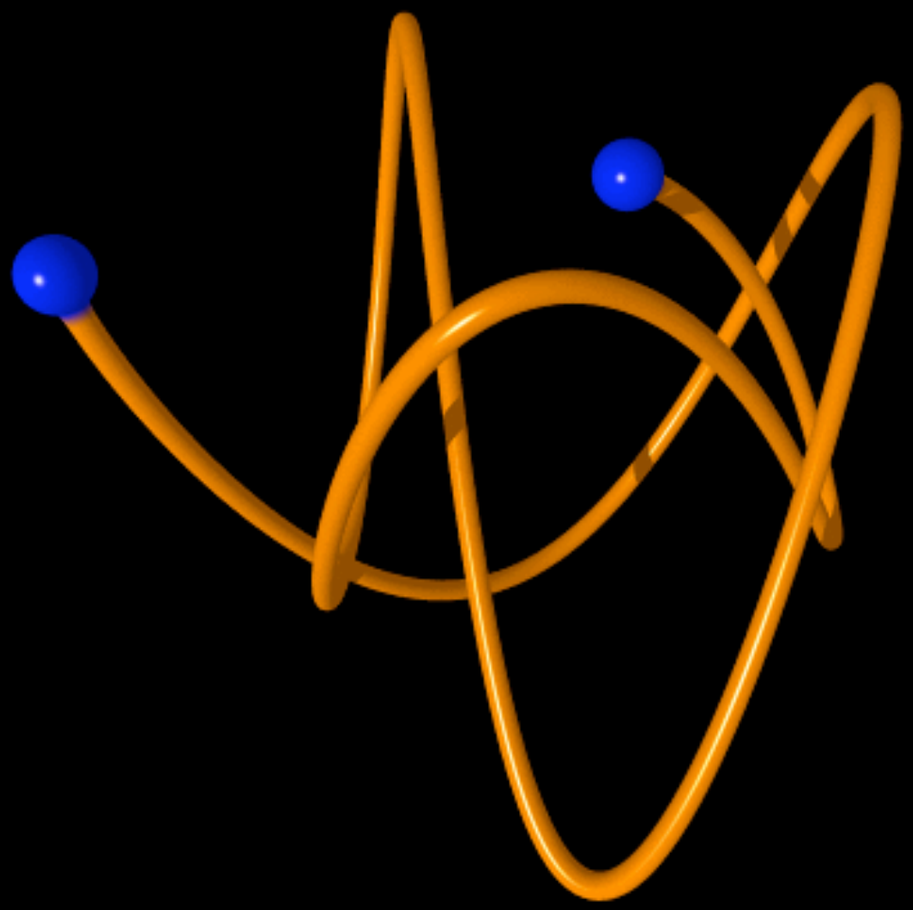


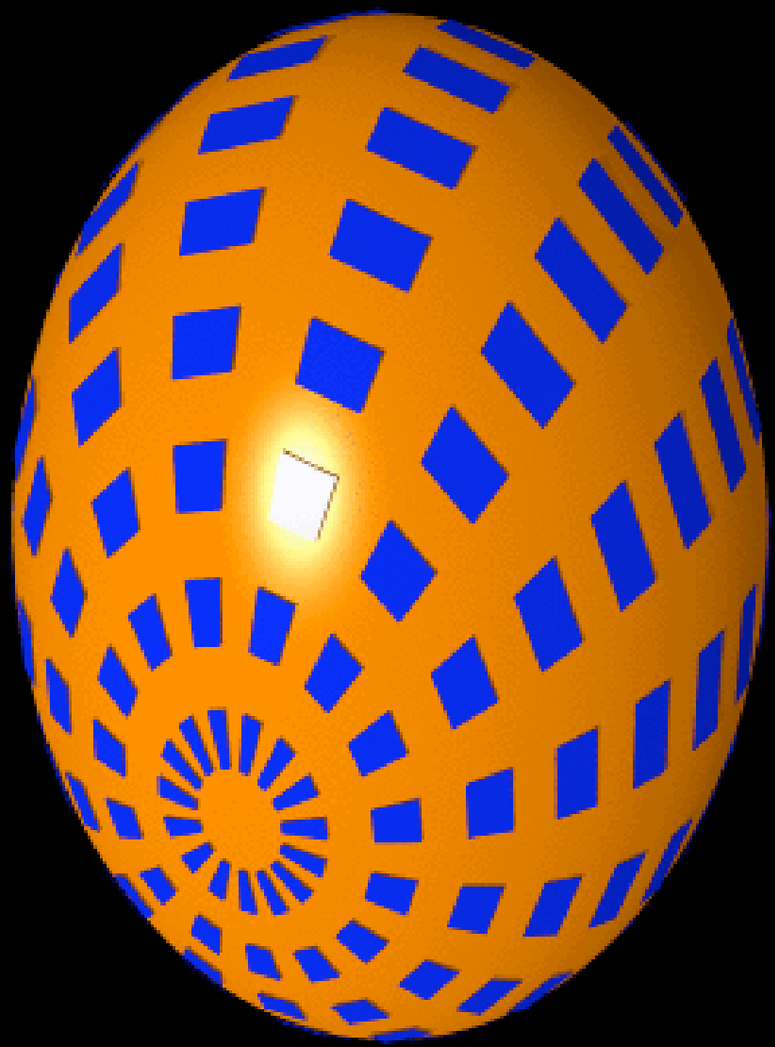


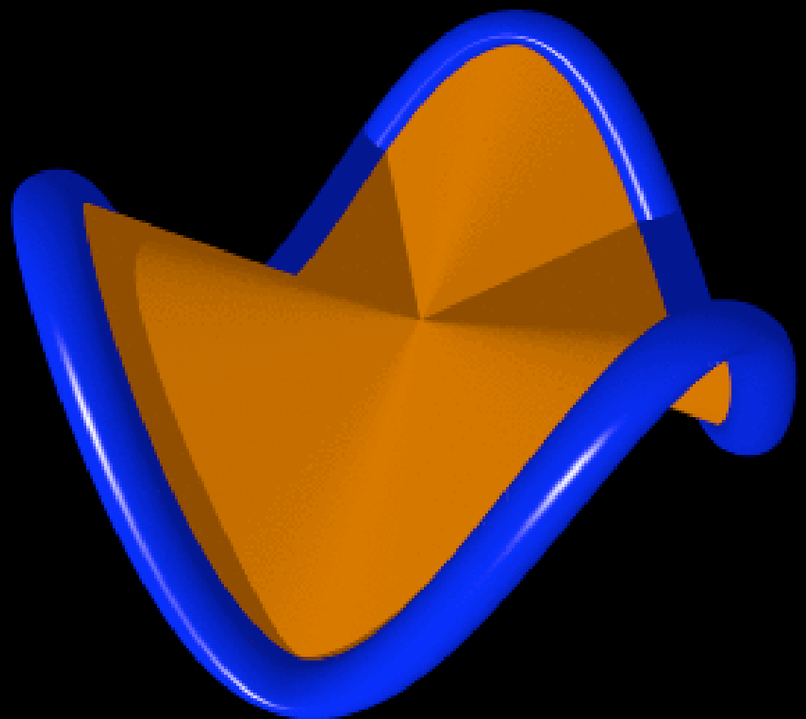


Simply connected ?

Boundary







That's all Folks!