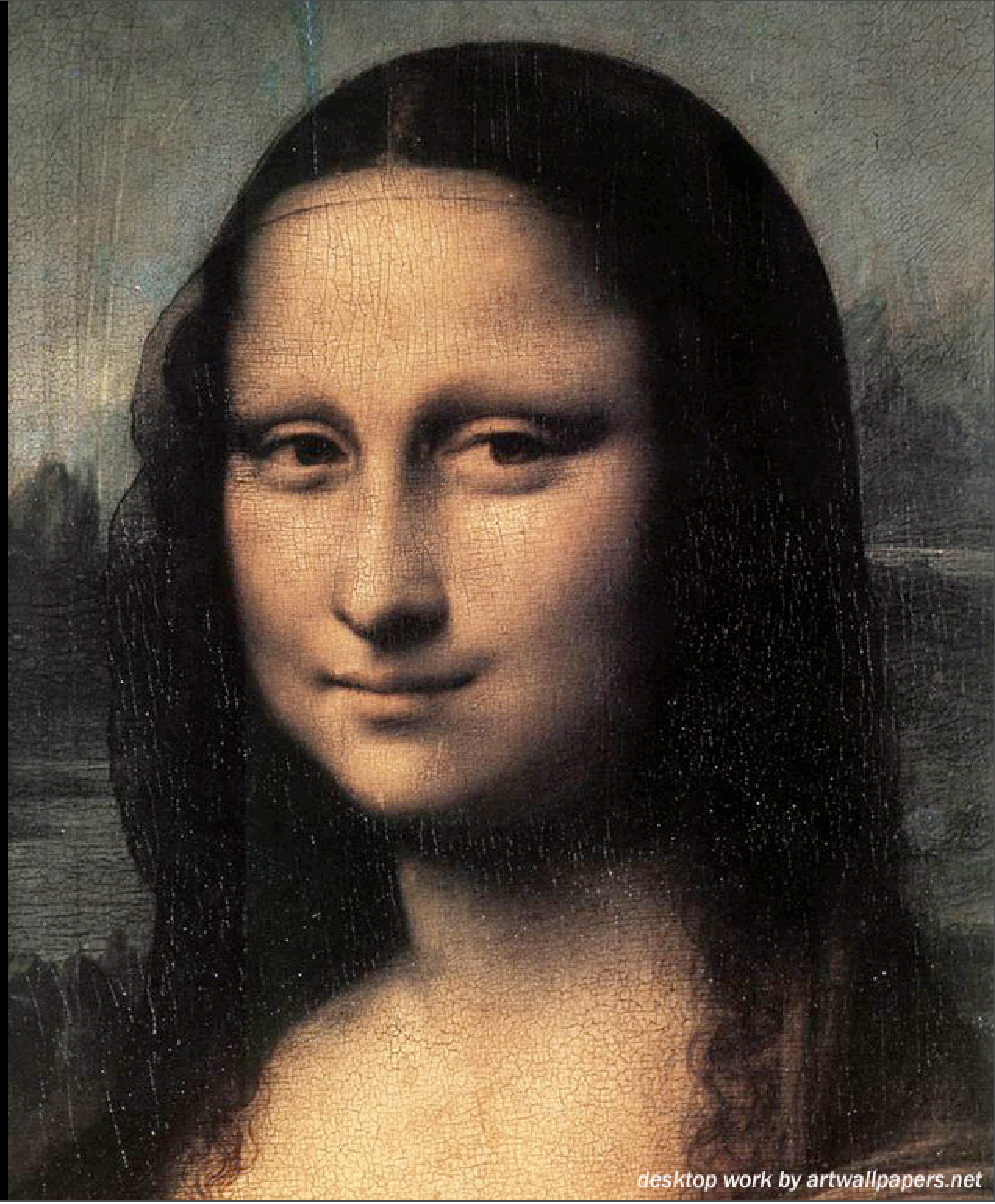
Math21a, Review Spring 2006, Part I

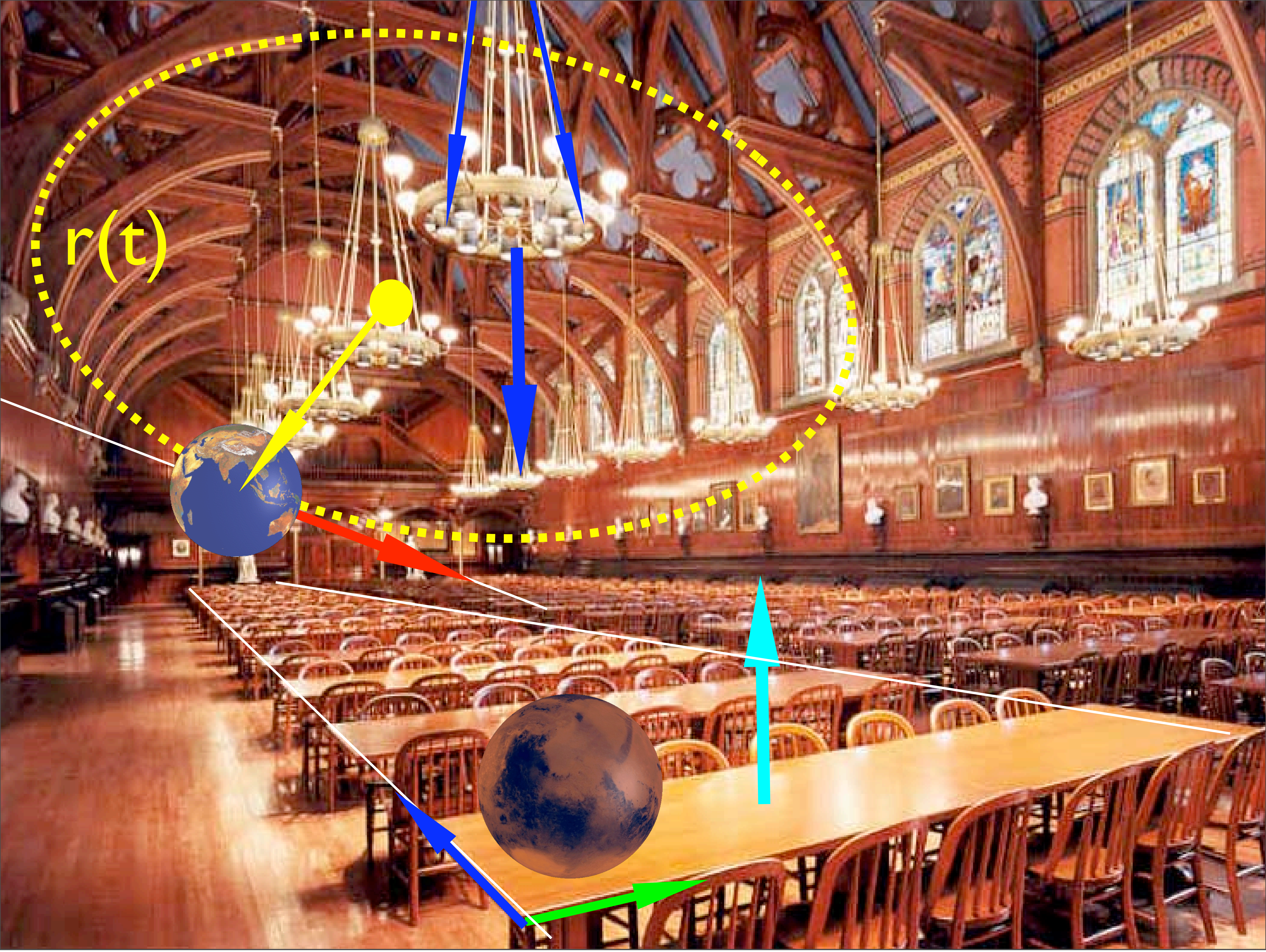
Oliver Knill, May 2006



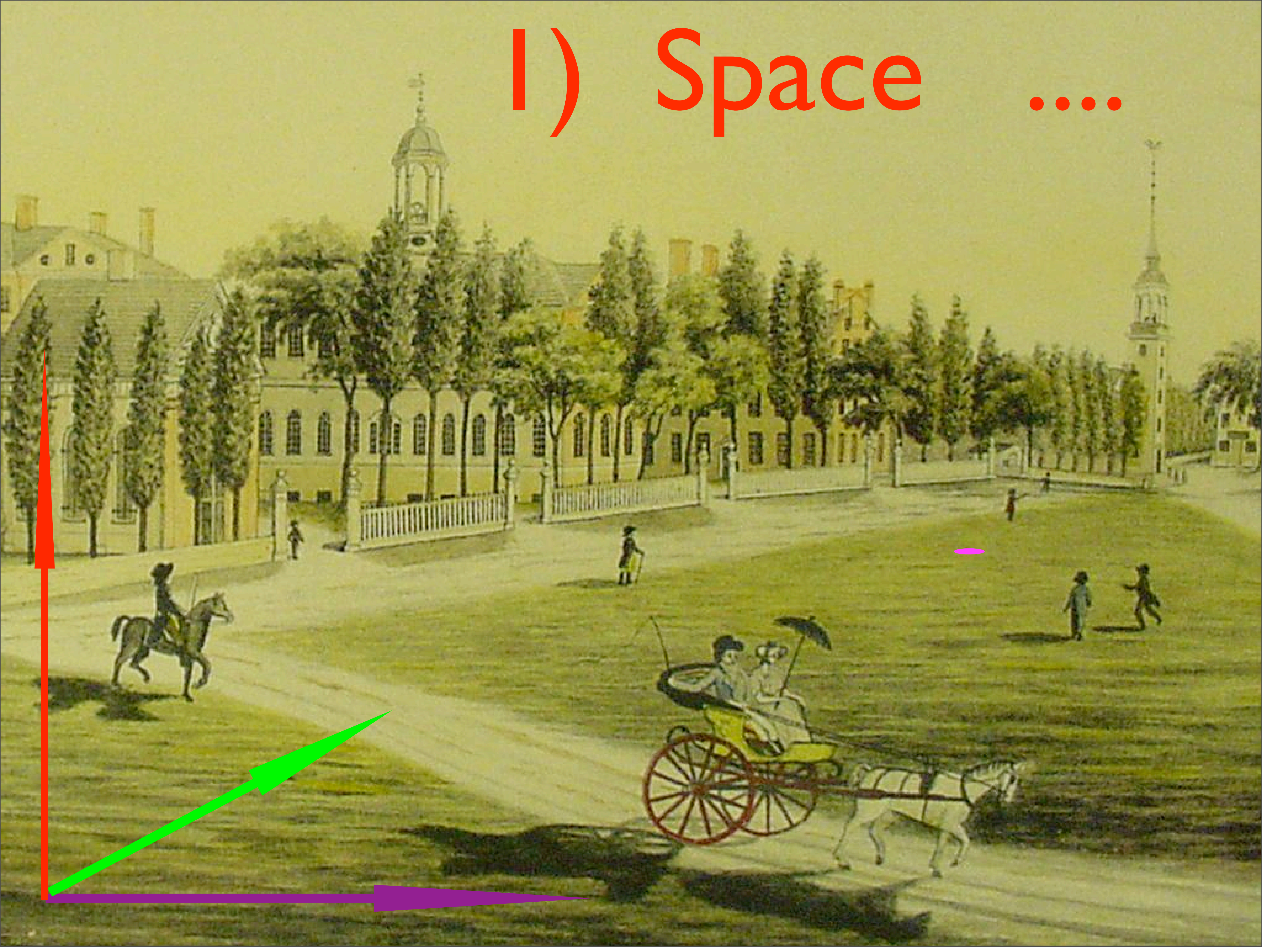
Leonardo da Vinci Mona Lisa

Part I: Geometry

Da Vinci: 1452-1519
Harvard college founded: 1626


r(t)

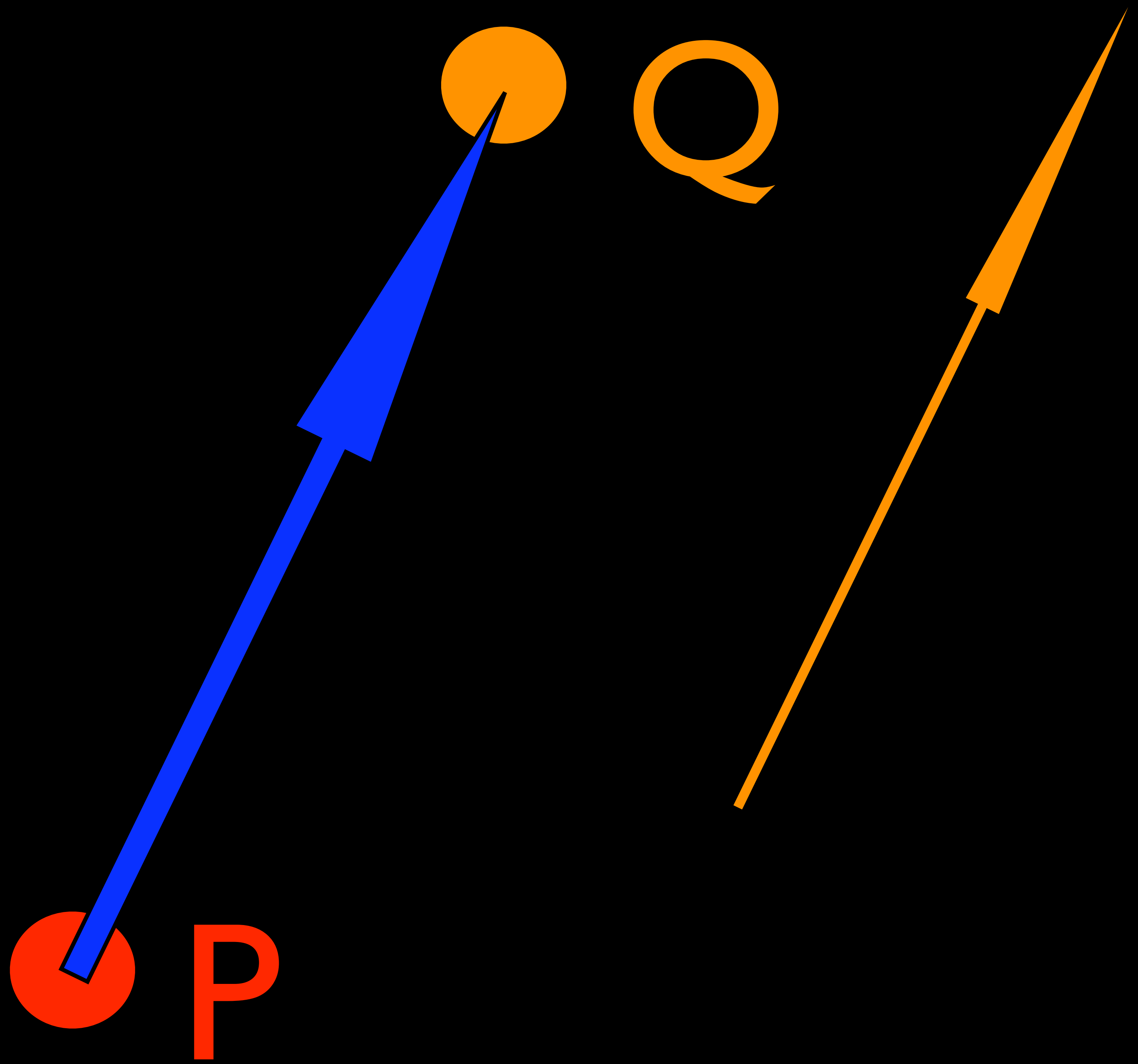
1) Space ....



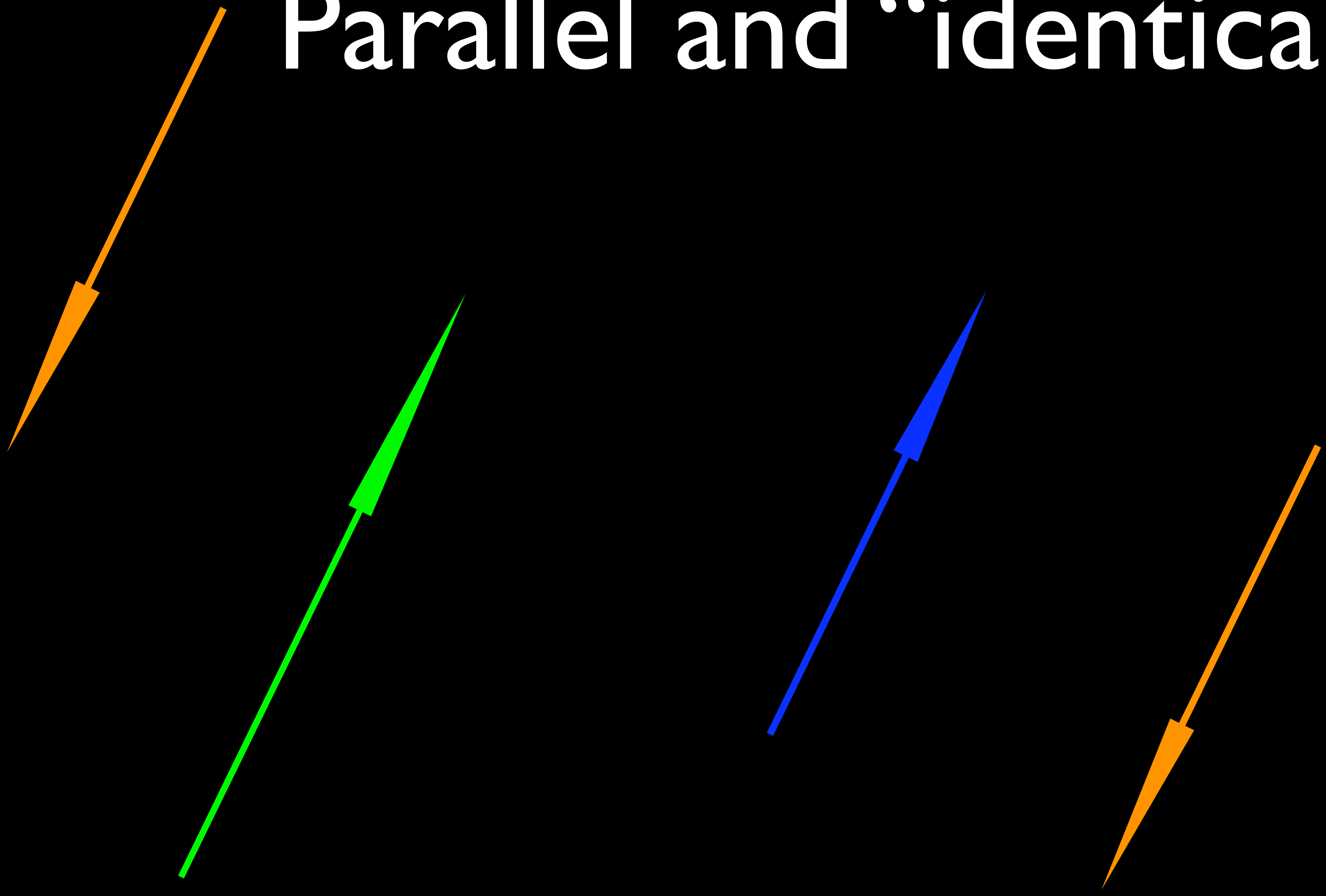

Lt. Govr. Oliver
Mr. Fairwather
Judge Lee
Judge Sewall
Col. Vaffel
Church
Common
Mr. Foxcr
Harvard College
CAMBRIDGE

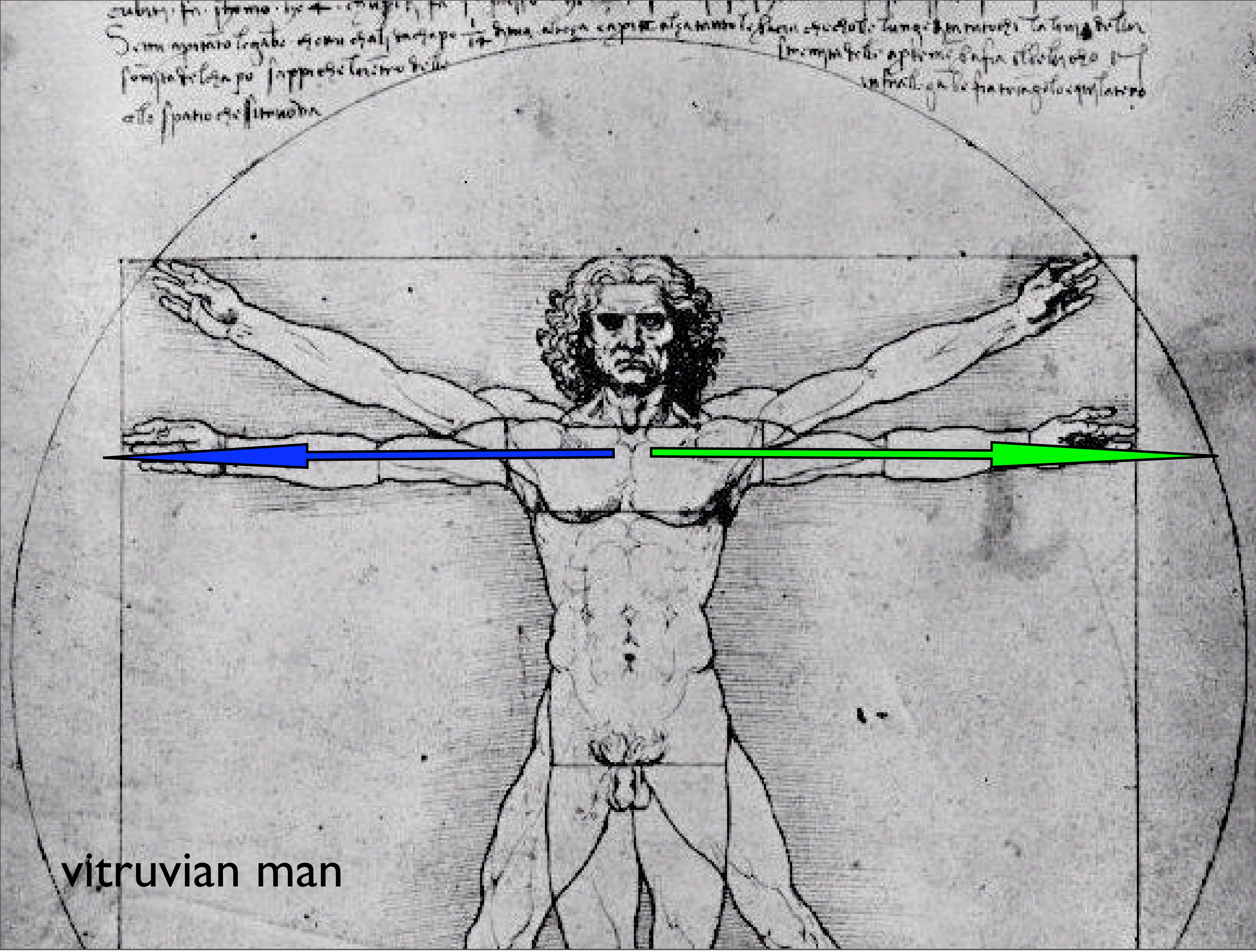
Vectors

Determined by two points P,Q.
Can be placed anywhere in space but translated vectors are considered "equal"



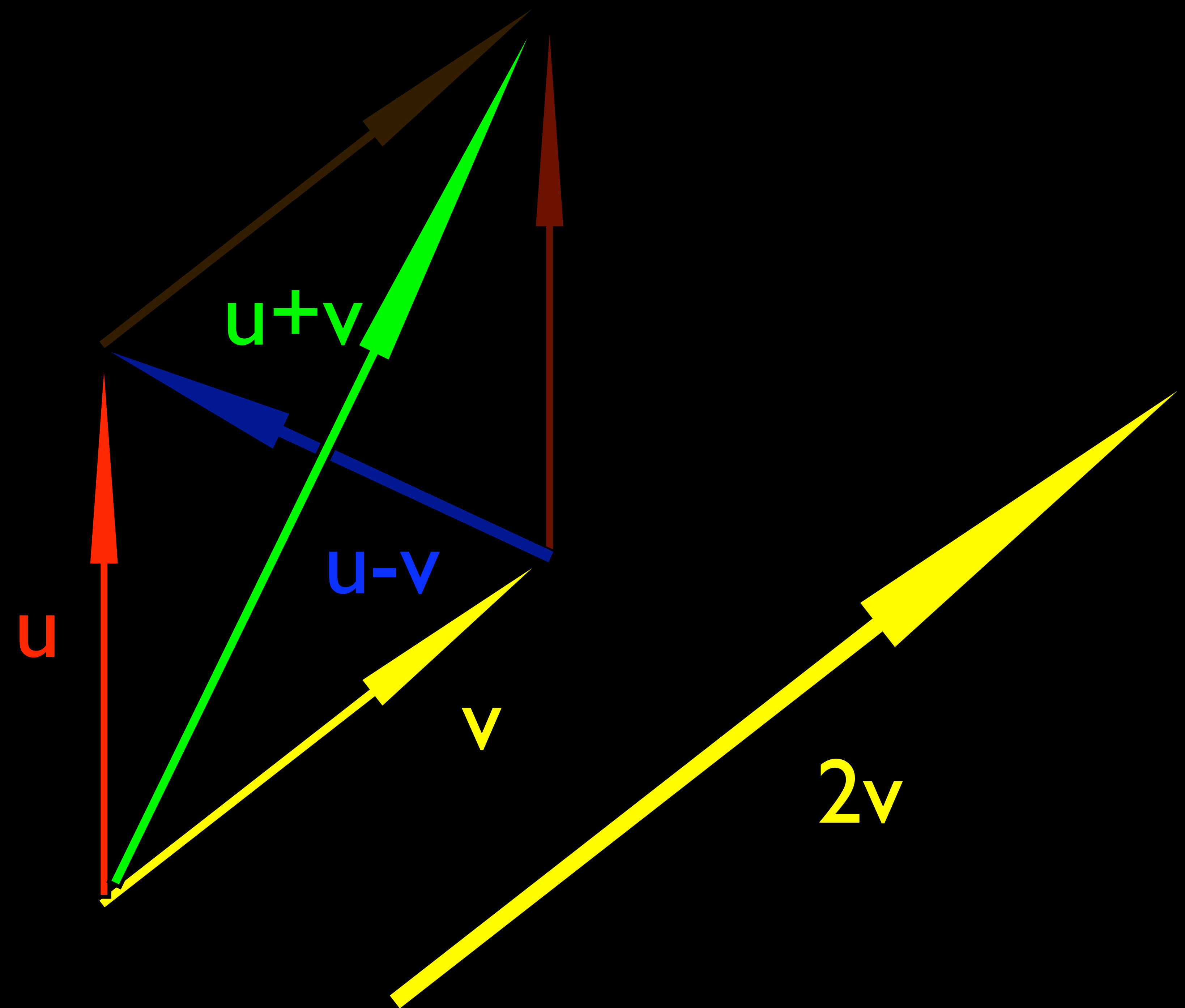
Parallel and “identical”

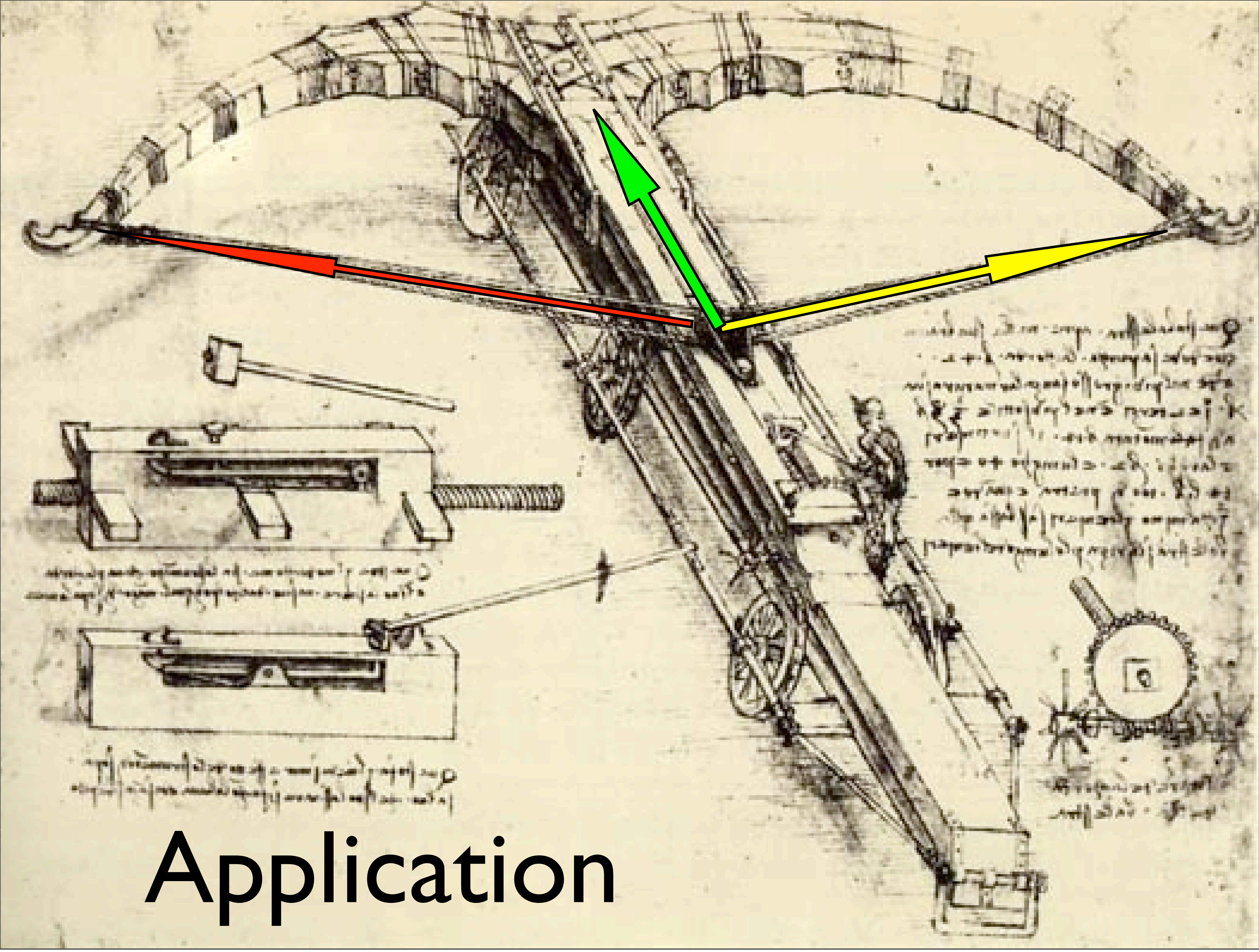




vitruvian man

Addition, Scaling

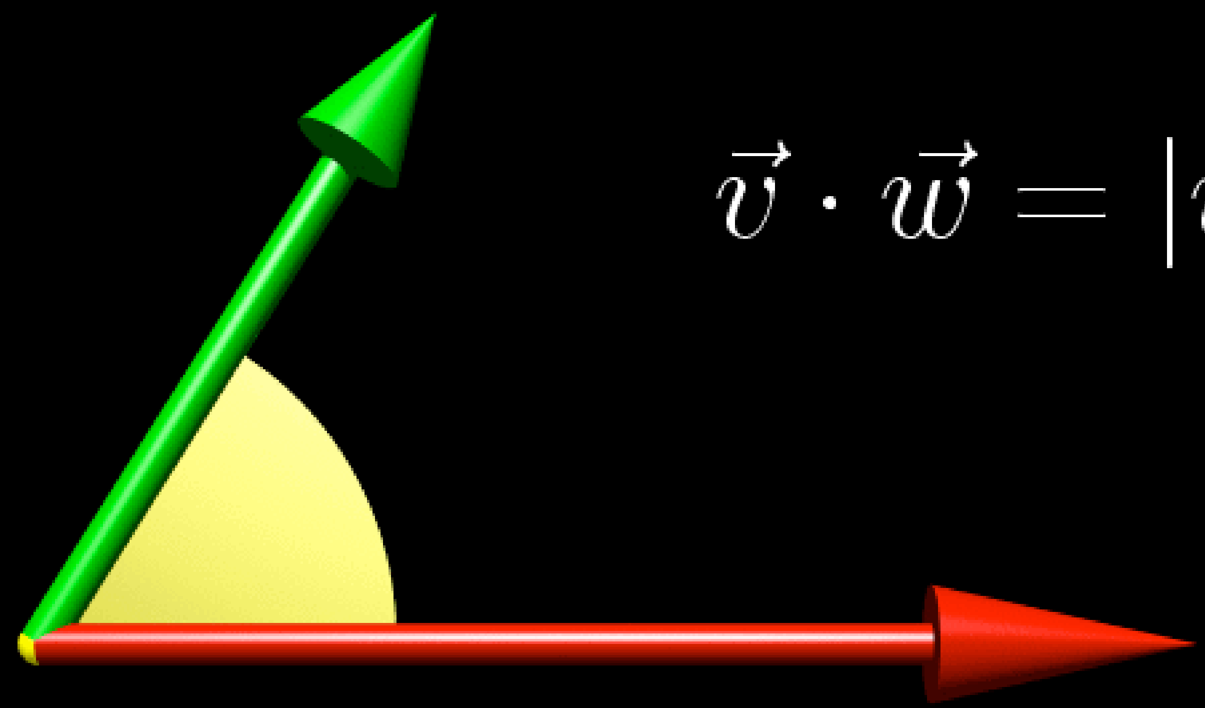




Application

Dot Product

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

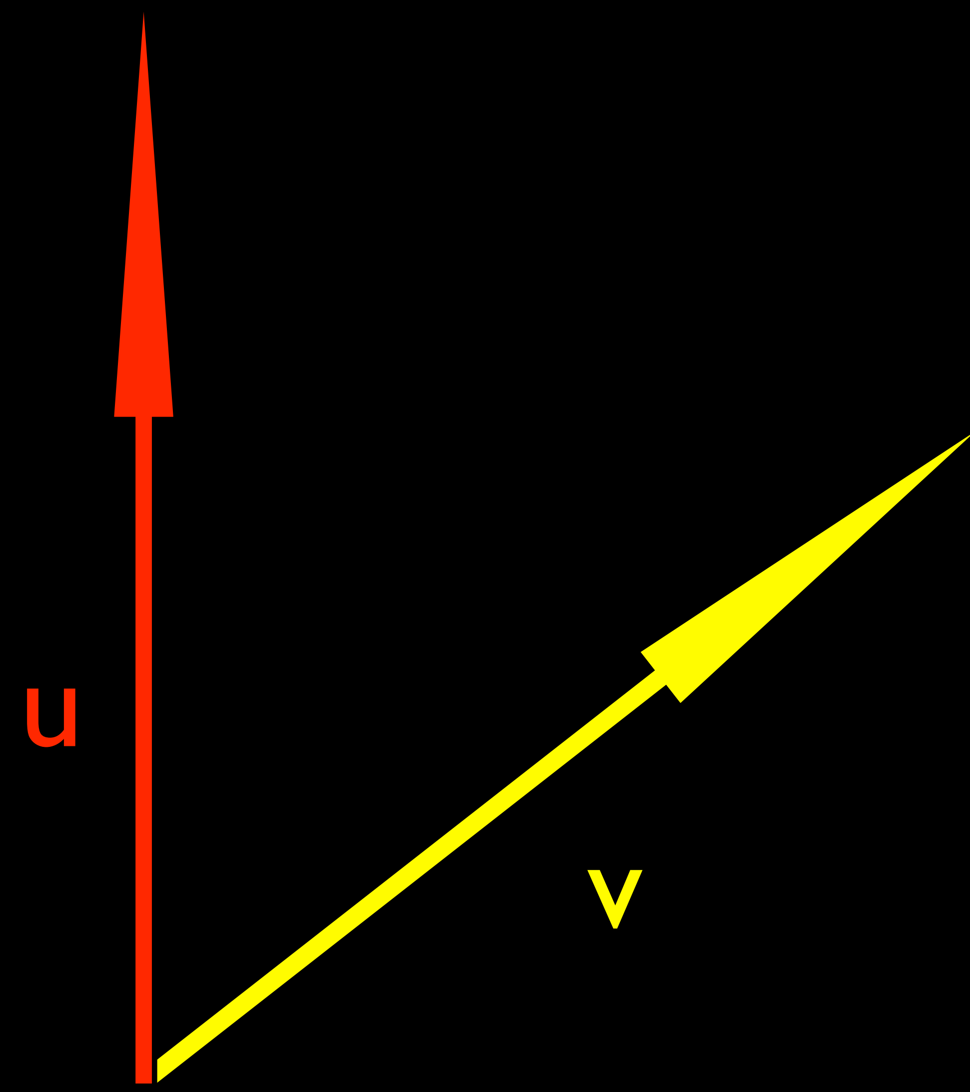


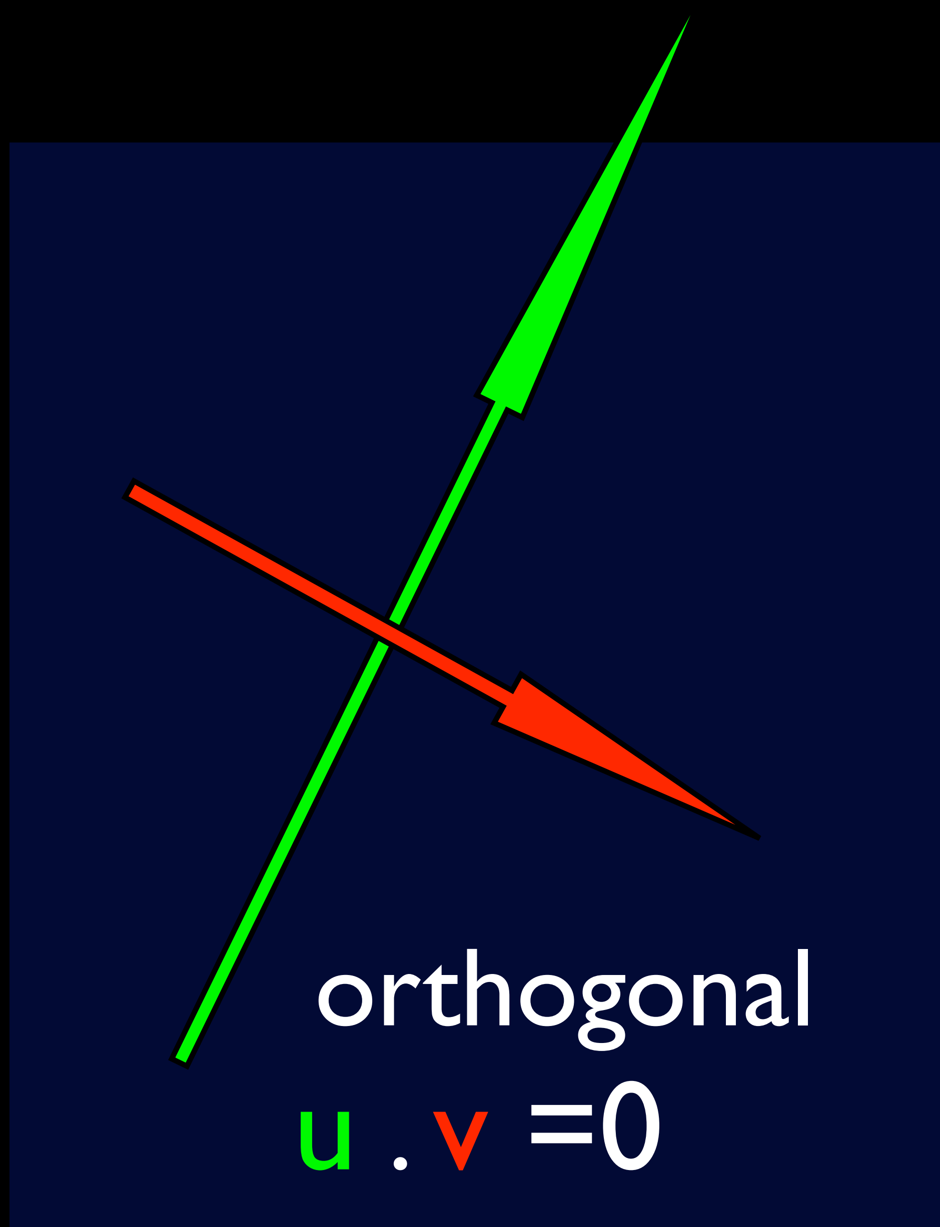
$$\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}|\ \cos(\theta)$$

Angle, Length, Orthogonality

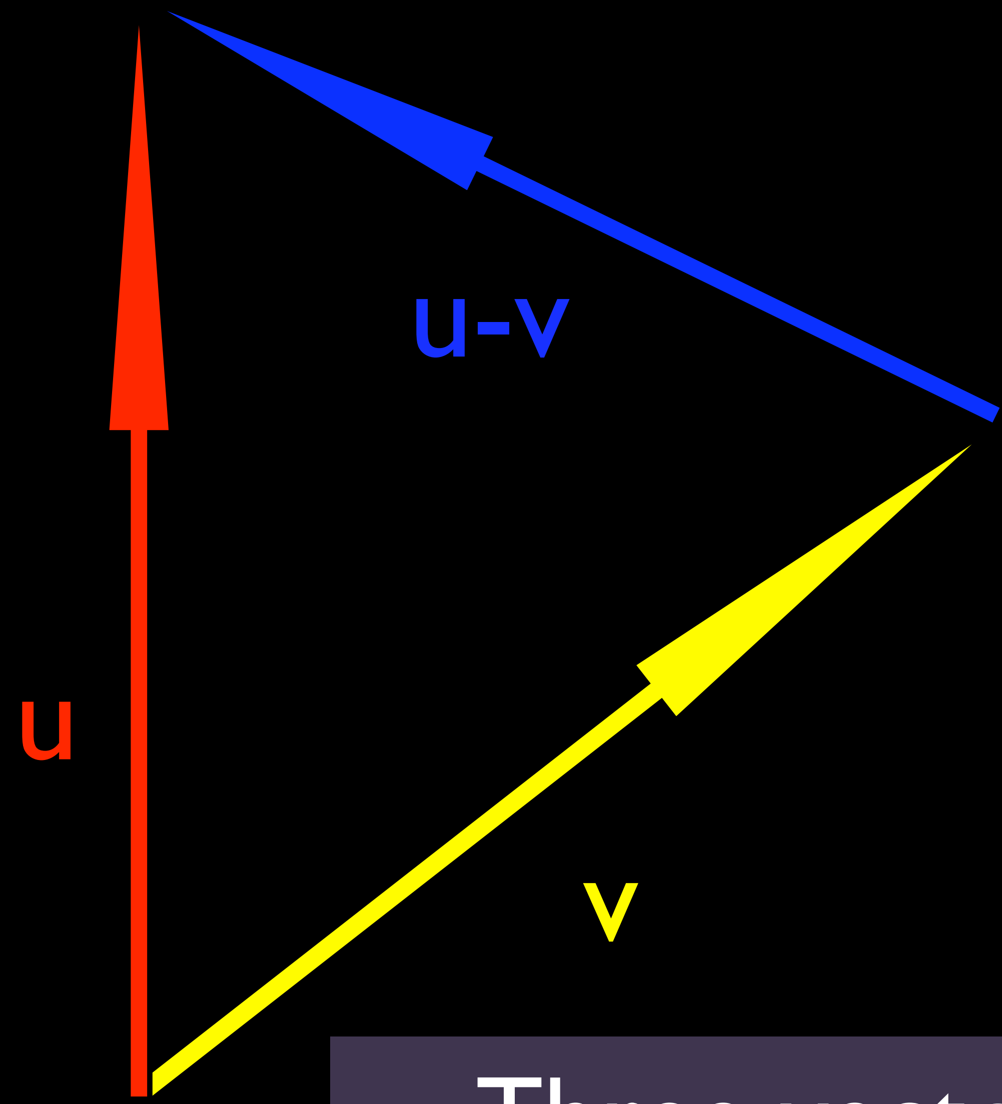
$\cos(t) = \mathbf{u} \cdot \mathbf{v} / (|u|\ |v|)$

$u.u = |u|^2$



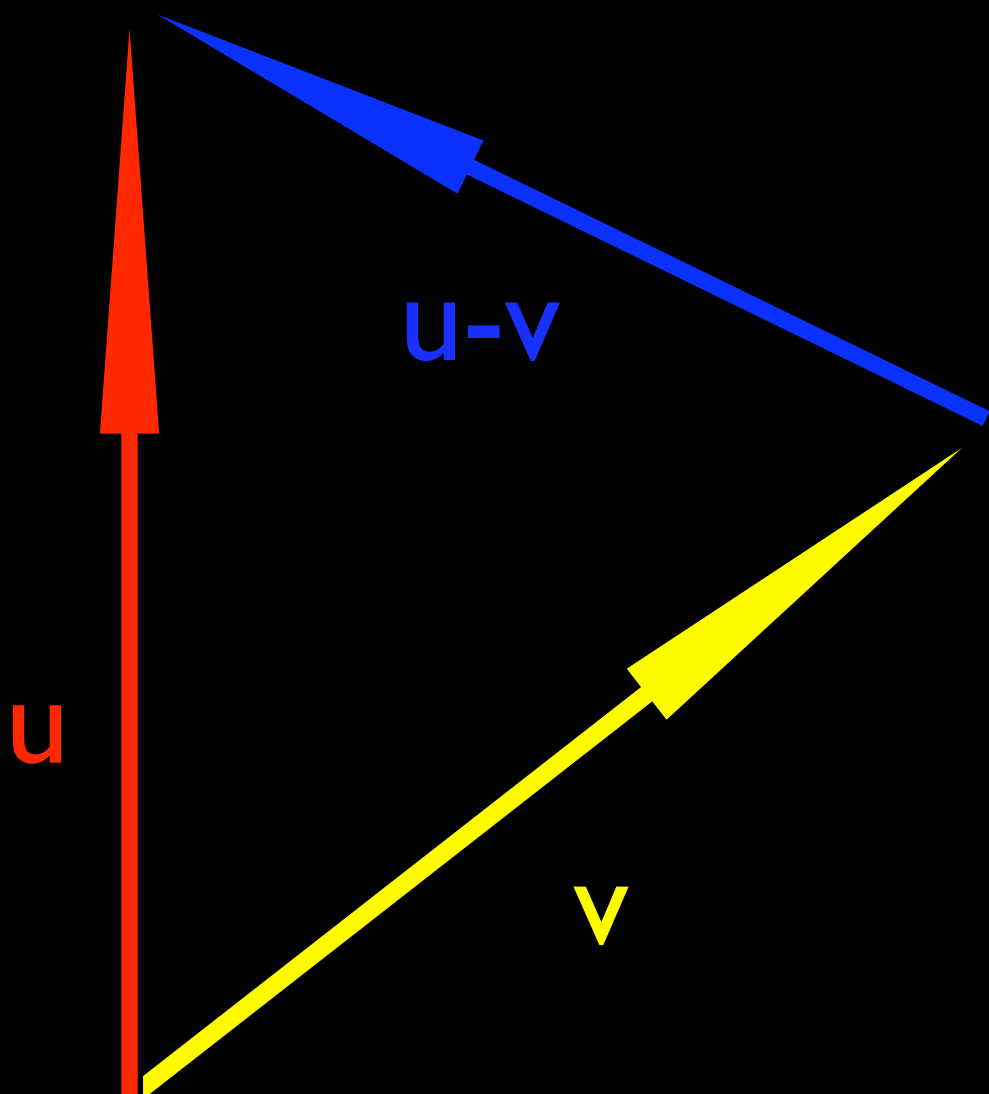


From vectors to angles



Three vectors u,v,u-v form a triangle.
We know |u|,|v|,|u-v|.
This determines the angles of the triangle.

From



$$|(u-v)|^2 = (u-v).(u-v)$$

$$=|u|^2+|v|^2 - 2(u.v)$$

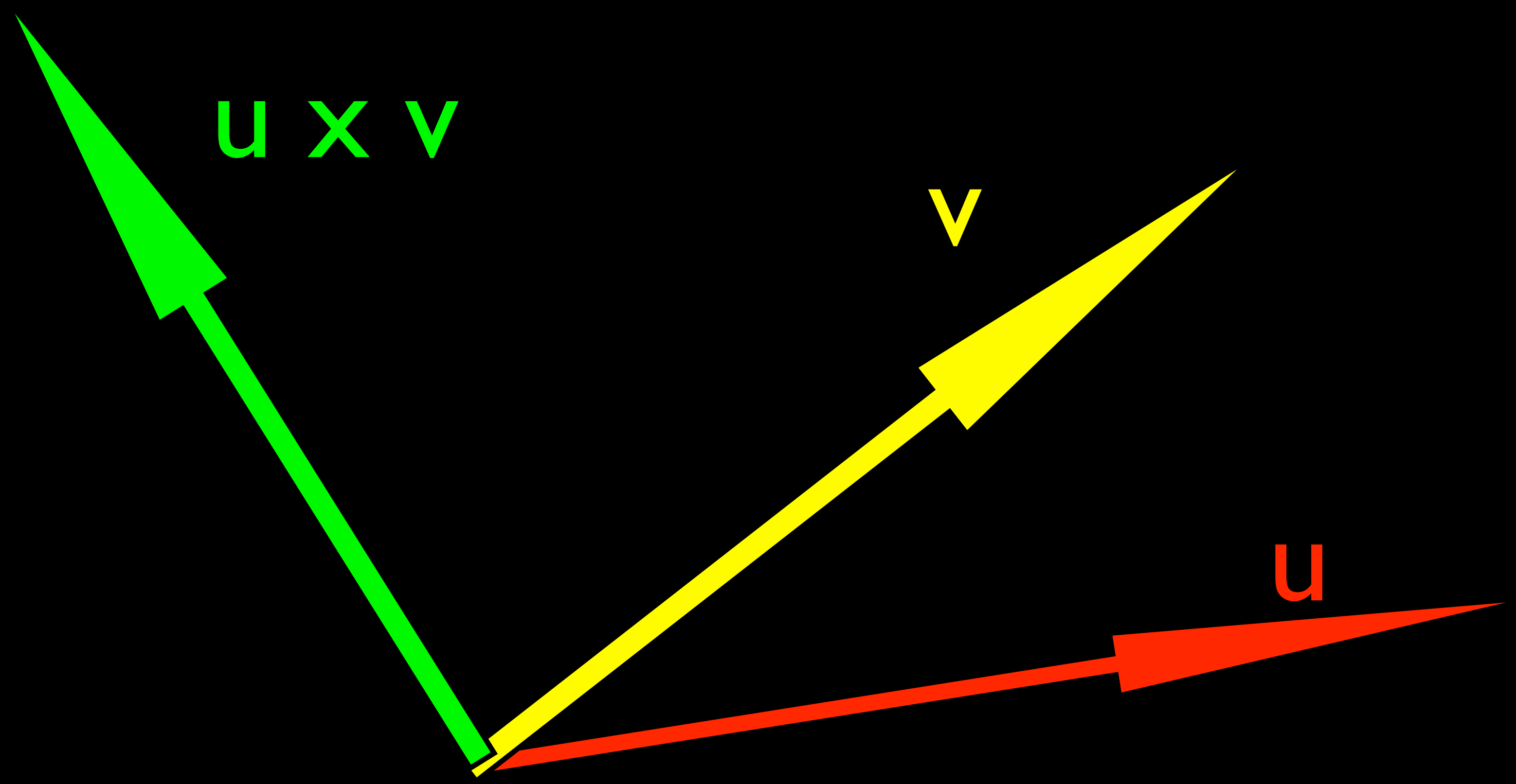
we know the dot product.

and so the angle.

$$\cos(t)=\frac{|(u-v)|^2 - |u|^2-|v|^2}{-2\,|u|\,|v|}$$

Cross Product

$$\vec{v} \cdot \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



Cross Product

$$\vec{v} \cdot \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} =$$

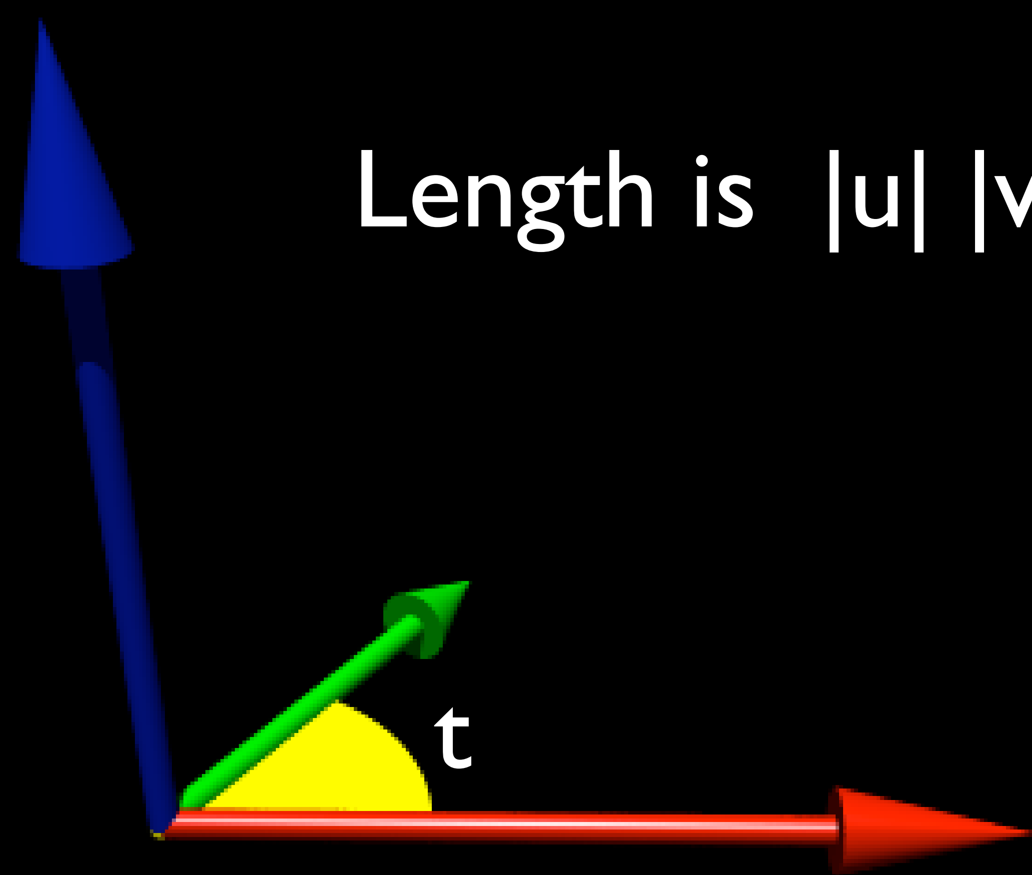
$$+ \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

$$- \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

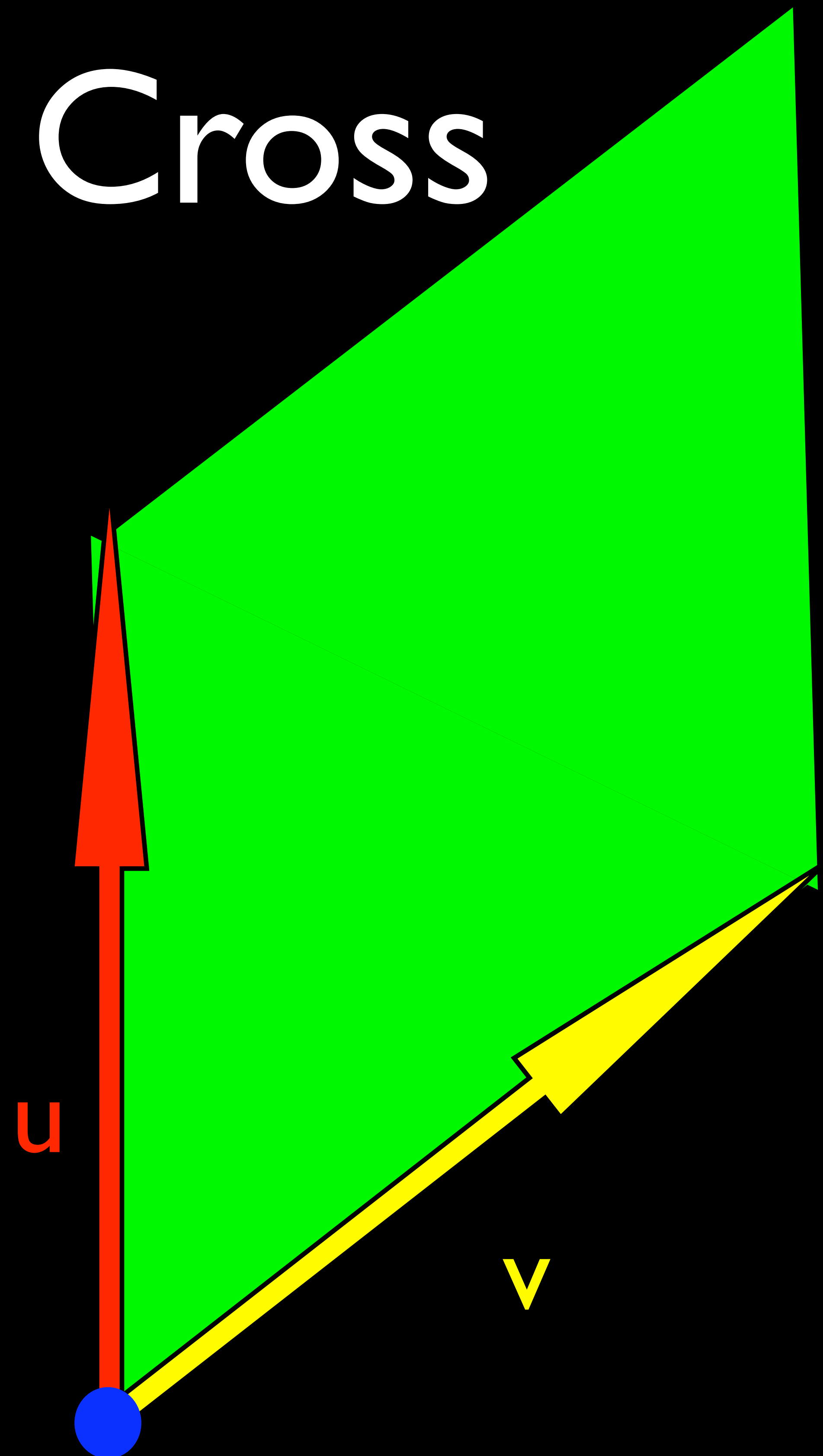
$$+ \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Length of Cross Product

Length is |u| |v| sin(t)



Cross Product and area.

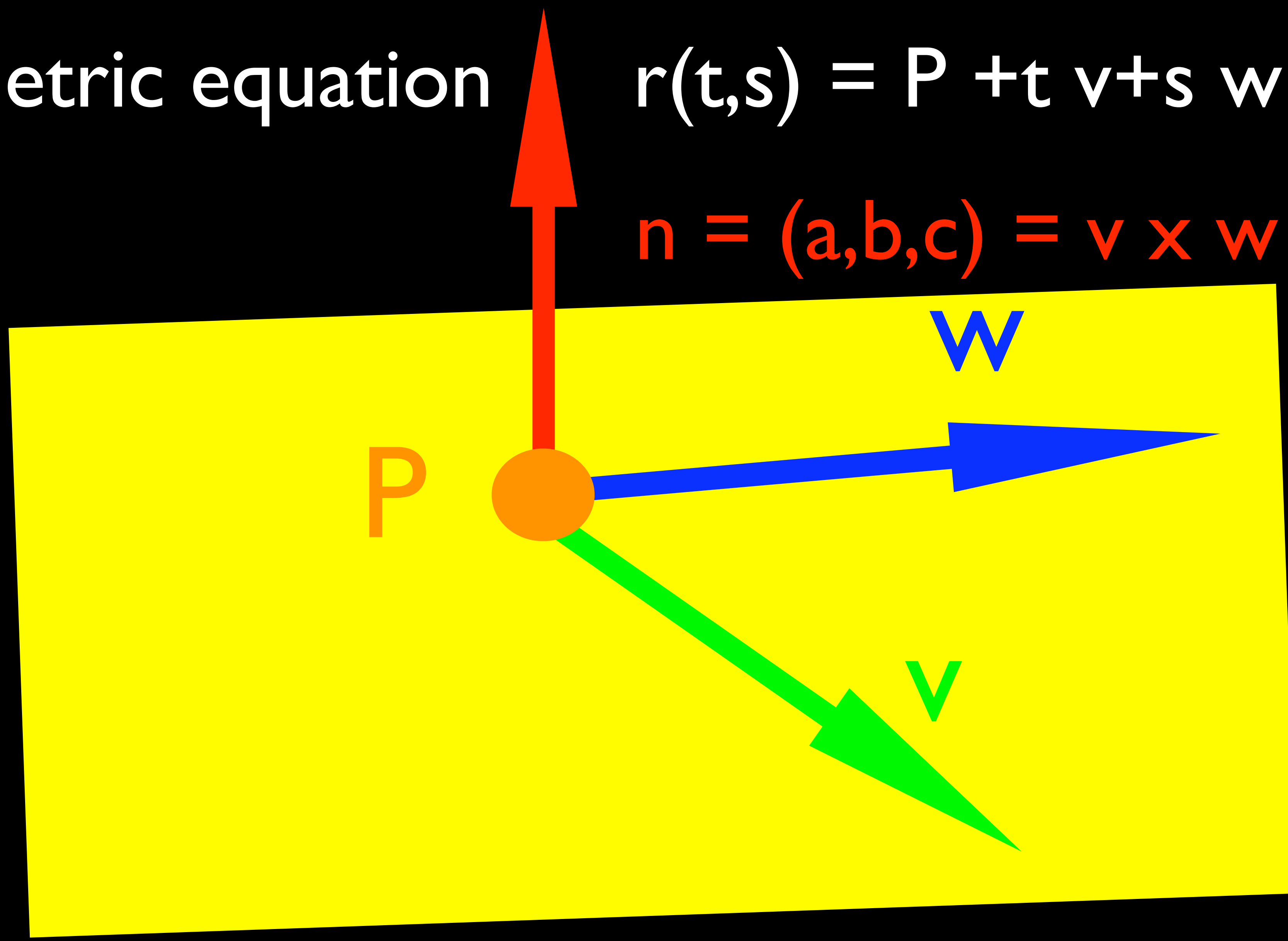


$$| \mathbf{u} \times \mathbf{v} | = |u|\ |v| \sin(t)$$

Planes

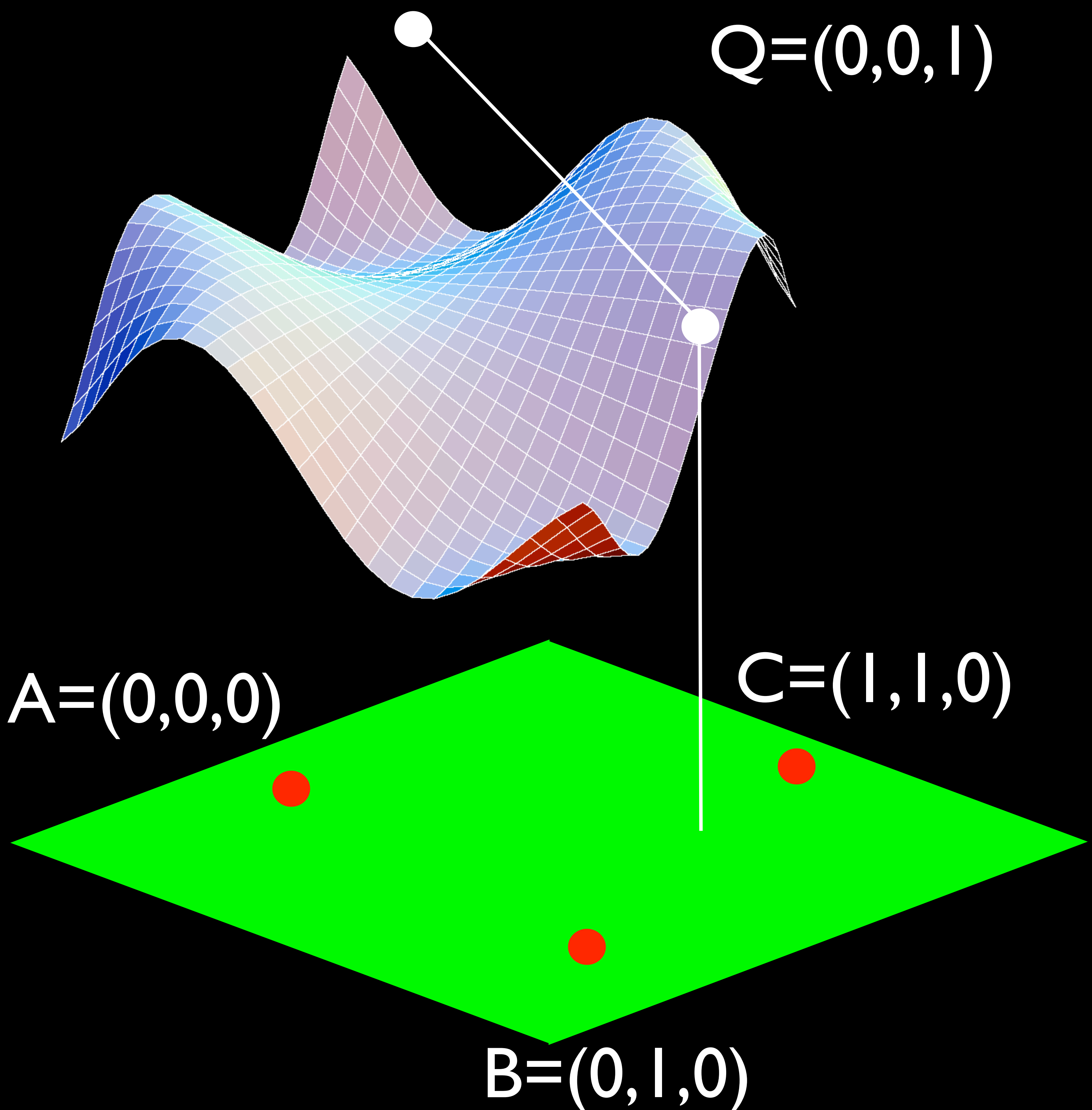
Parametric equation $r(t,s) = P + t\,v + s\,w$

$n = (a,b,c) = v \times w$



$$a\,x + b\,y + c\,z = d$$

Problem 1

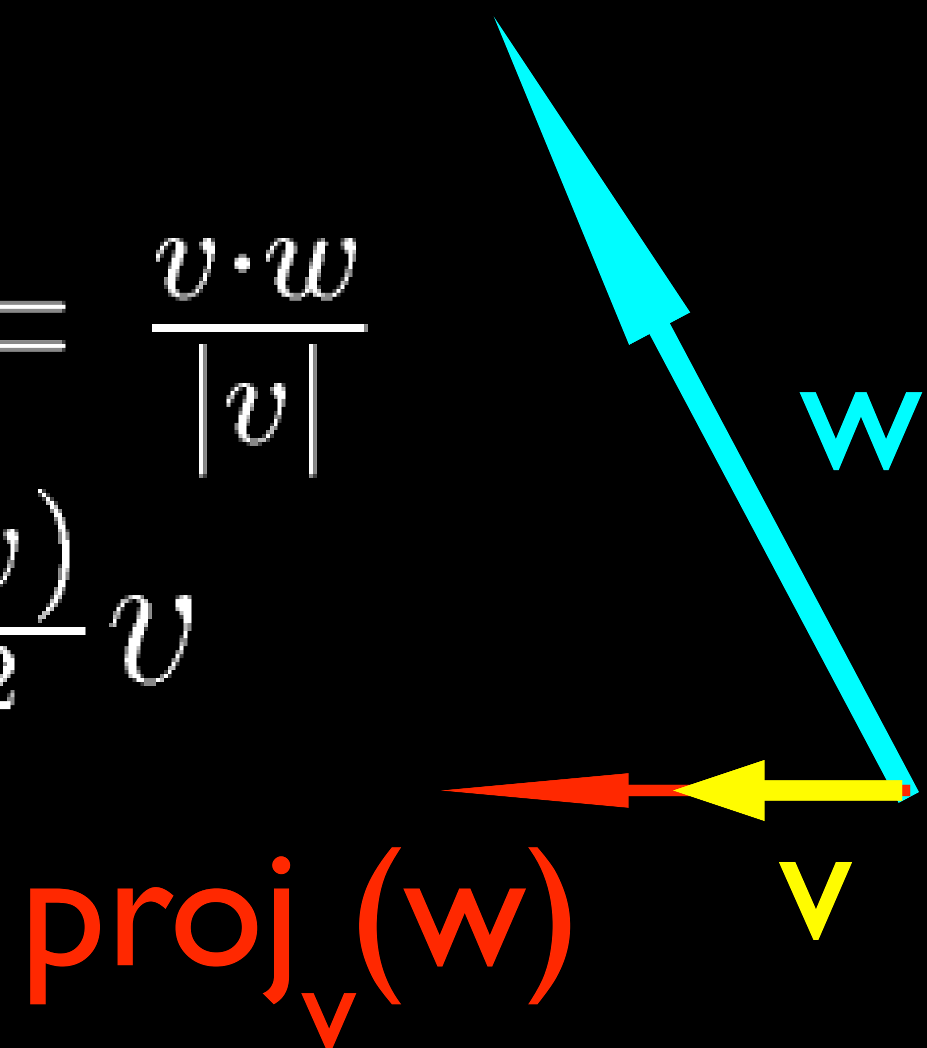


What are the points, for which the distance to Q is equal to the distance to the plane through A,B and C?

Vector and

$$\mathrm{comp}_v(w) = \frac{v \cdot w}{|v|}$$

$$\mathrm{proj}_v(w) = \frac{(v \cdot w)}{|v|^2} v$$



Example: find the vector projection of w onto v

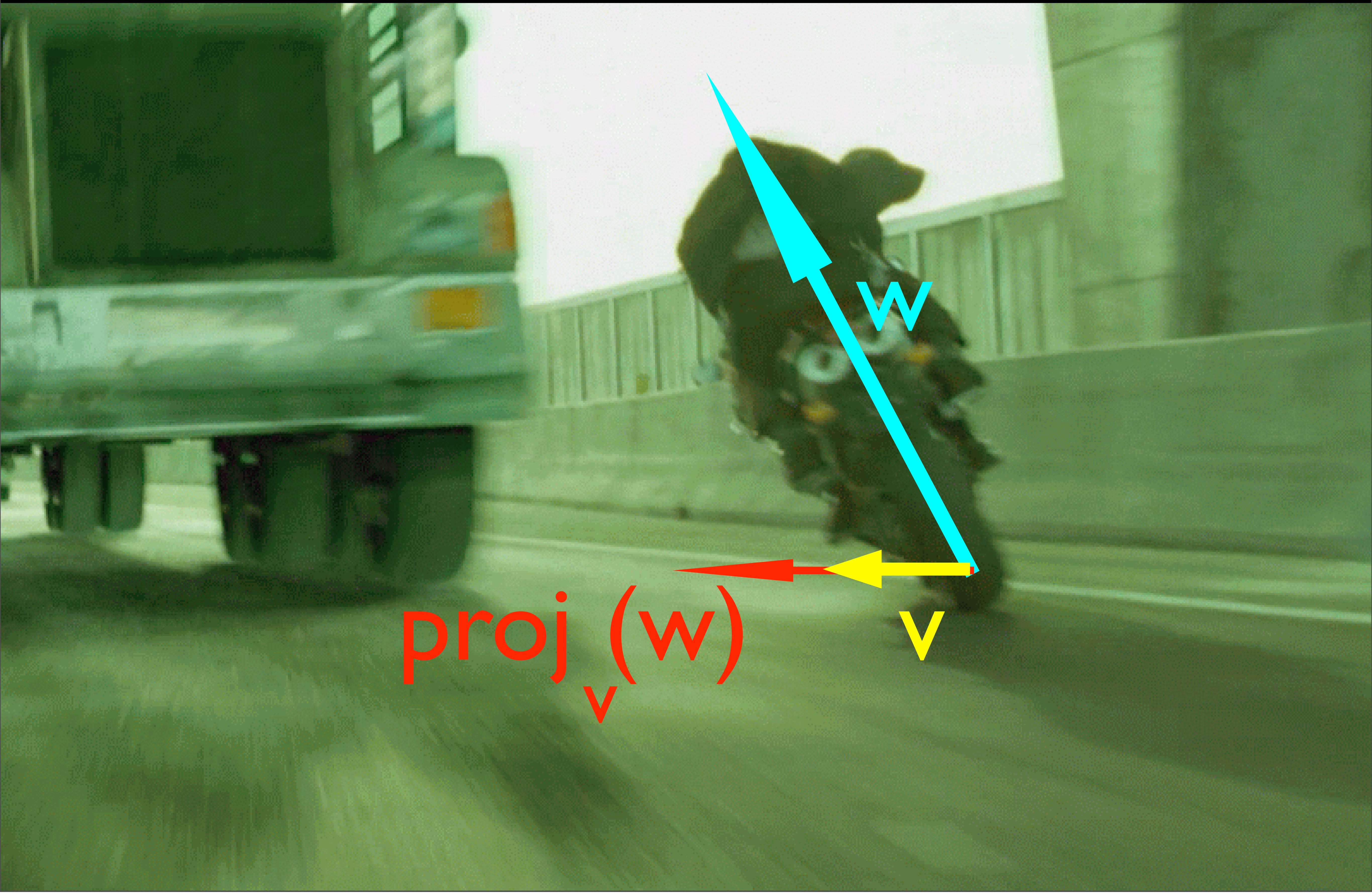
$v=(1,1,0)$ $w=(3,1,-5)$

The vector projection of w onto v is

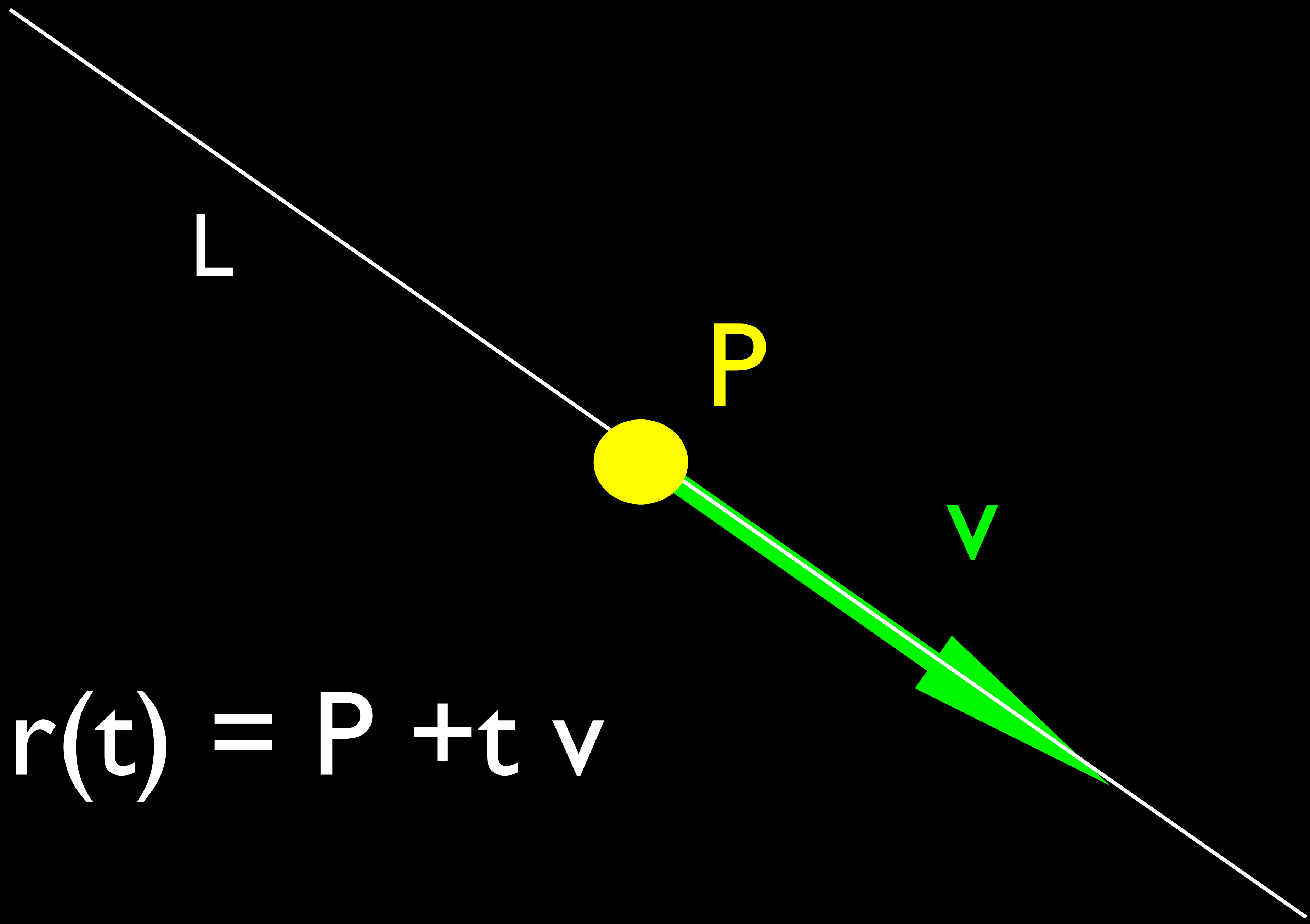
$4v/2 = 2v$

$\text{proj}_v(w)$

Projections

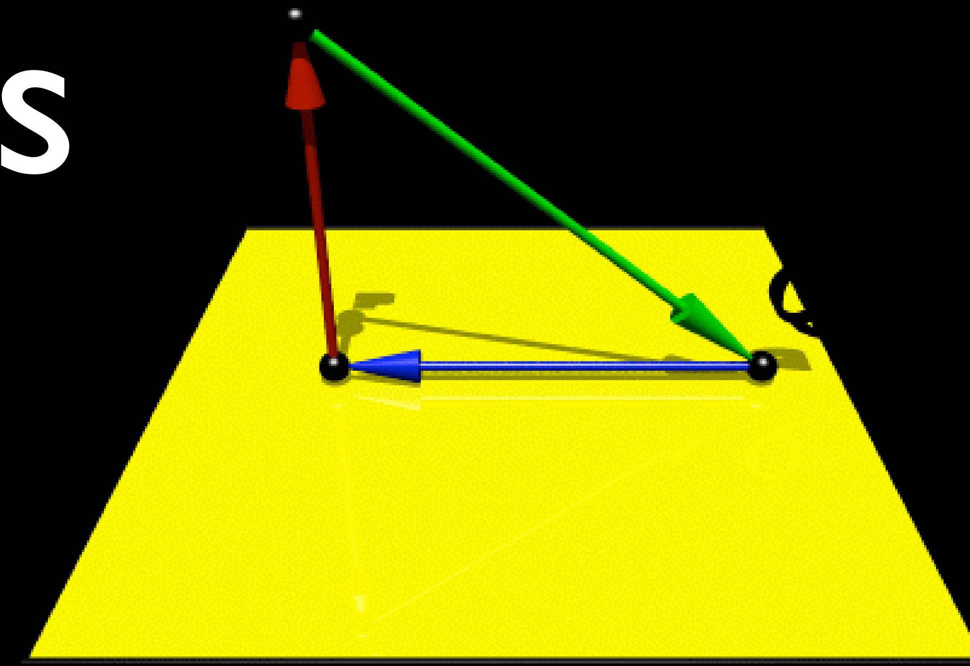


Parametrized Lines

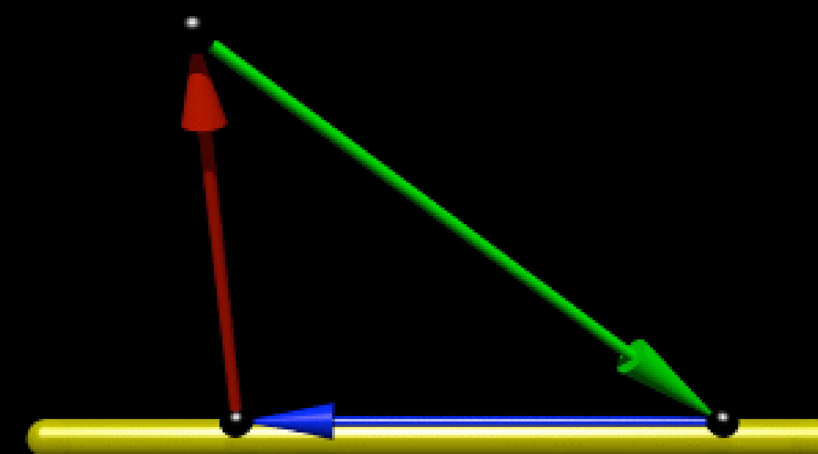


$r(t) = P + t\,v$

Distance Formulas

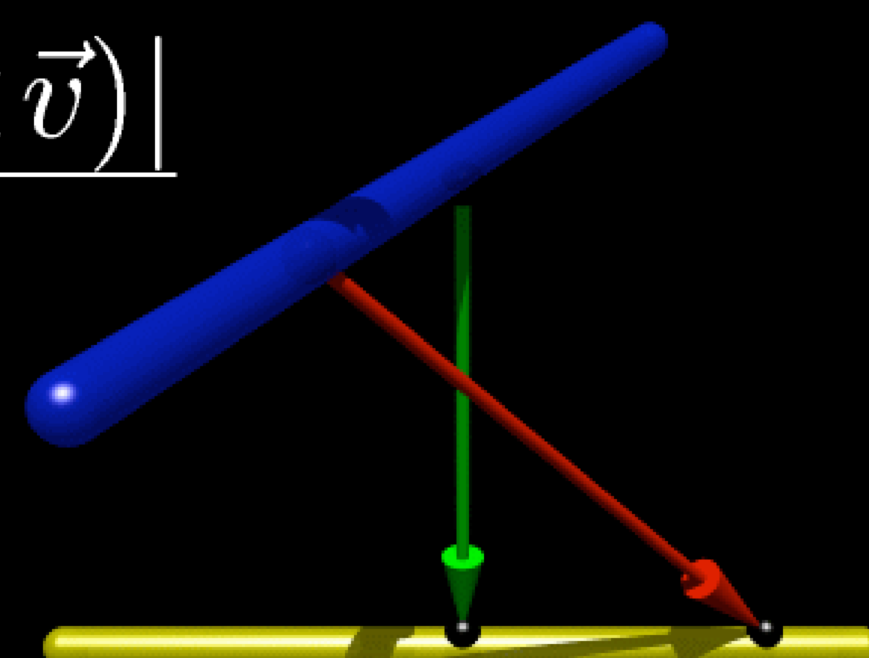


$$d(P,\Sigma) = \frac{|(\vec{PQ})\cdot\vec{n}|}{|\vec{n}|}$$

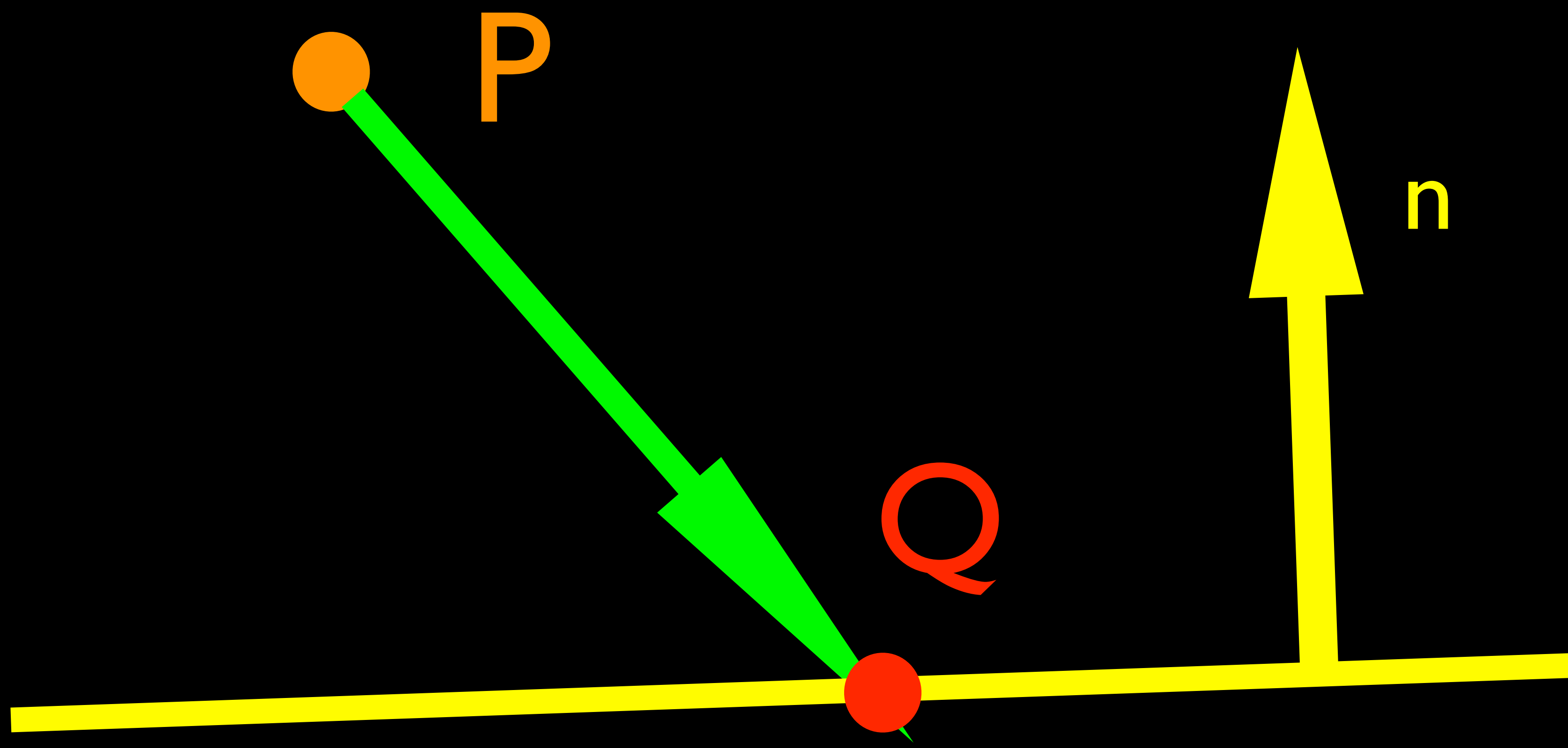


$$d(P,L) = \frac{|(\vec{PQ})\times\vec{u}|}{|\vec{u}|}$$

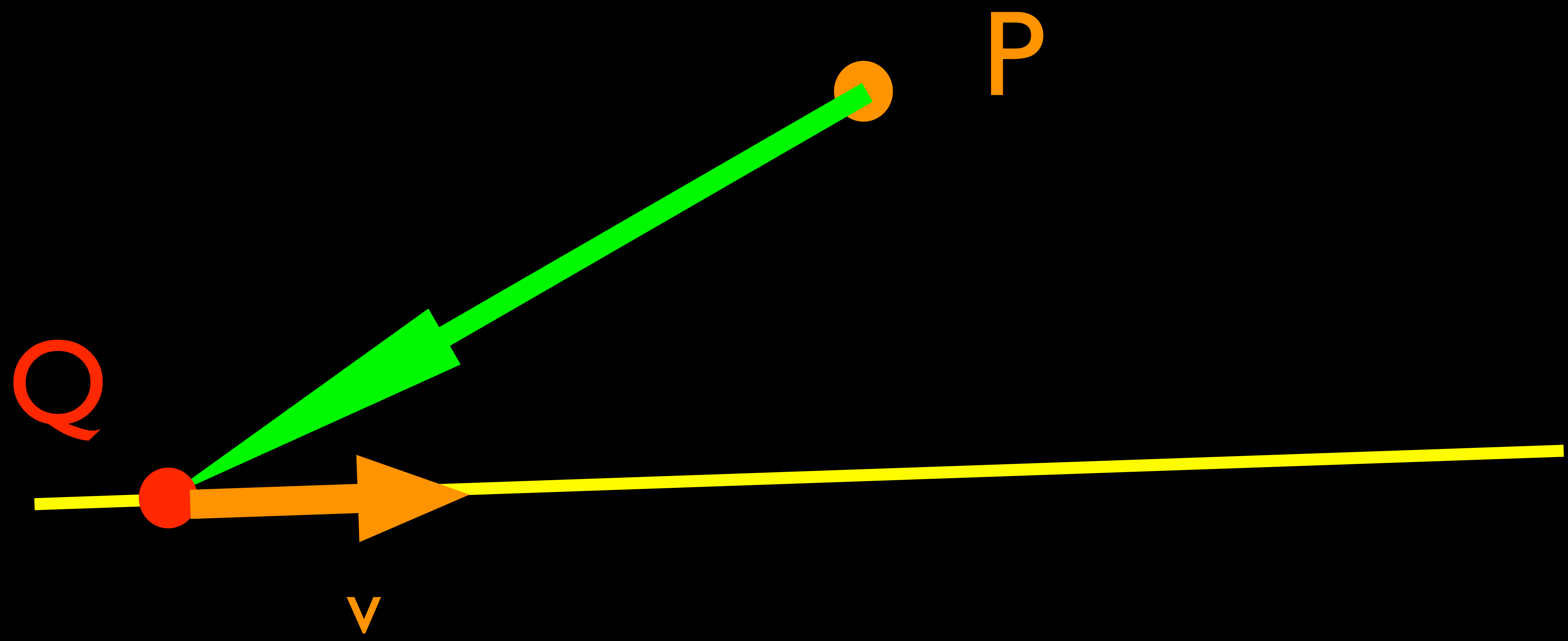
$$d(L,M) = \frac{|(\vec{PQ})\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$



Distance Point-Plane

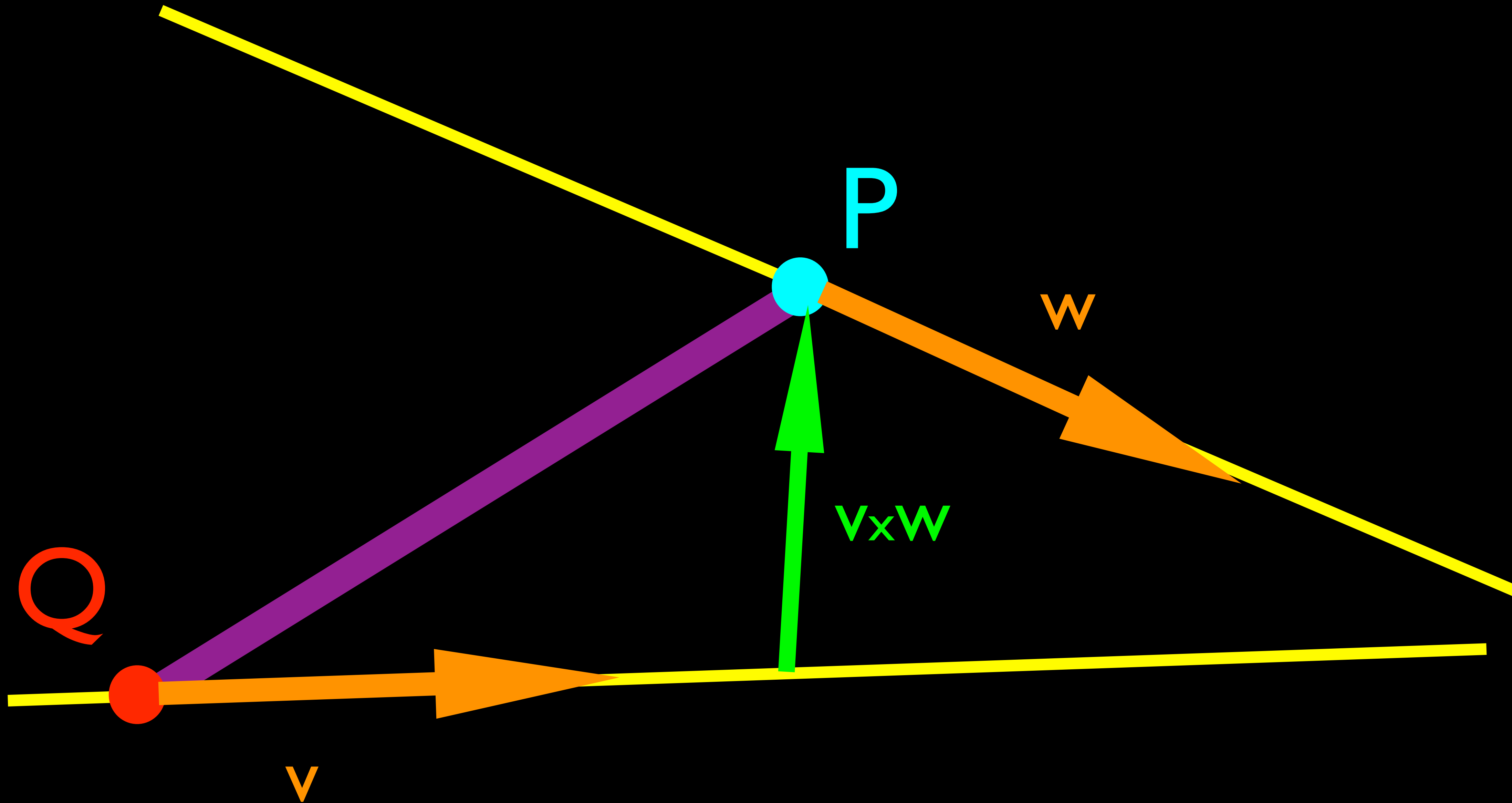


Distance Point-Line



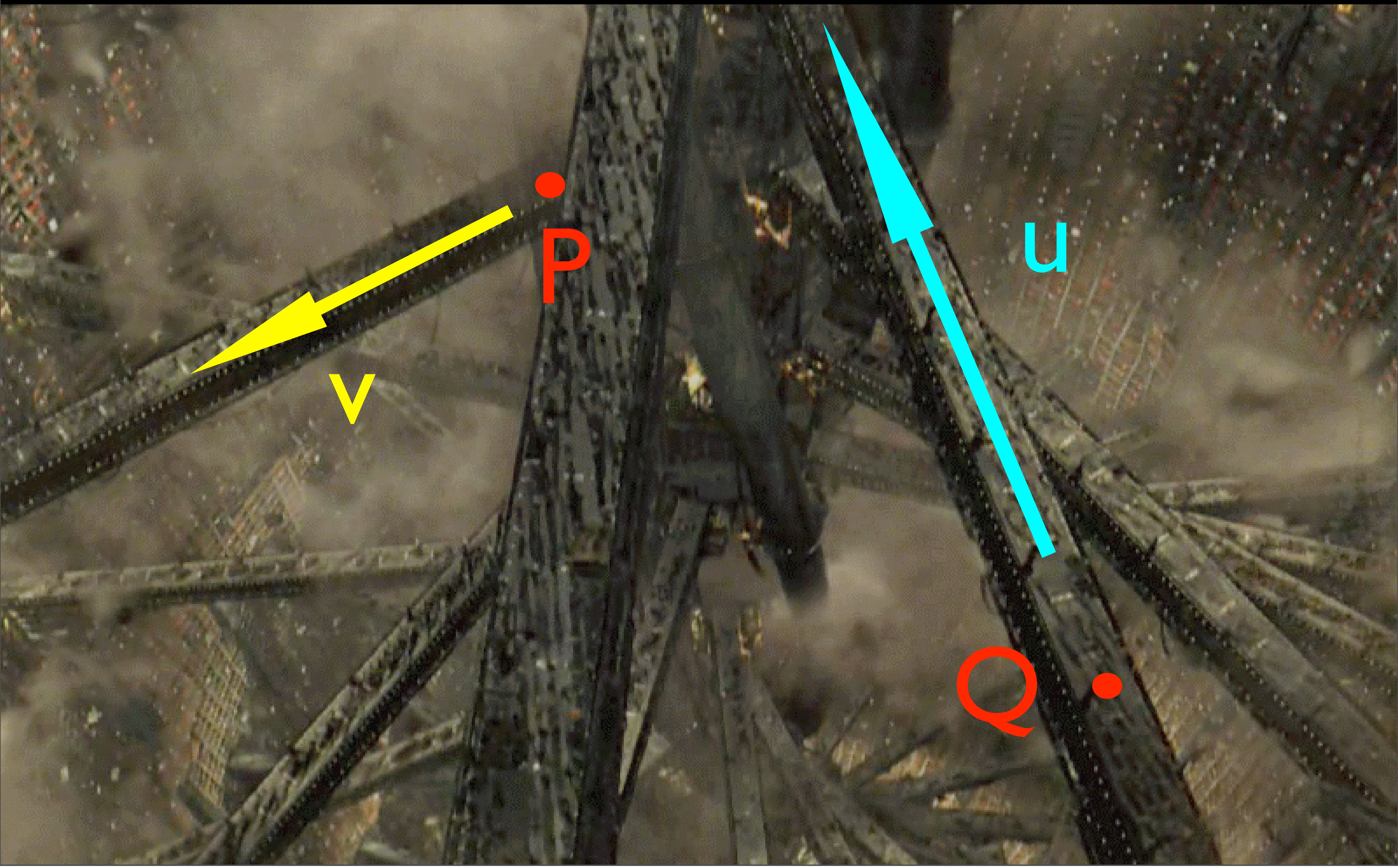
$$d = |\vec{PQ}|\sin(\theta) = \frac{|\vec{PQ}||\vec{v}|\sin(\theta)}{|\vec{v}|} = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$

Distance Line-Line

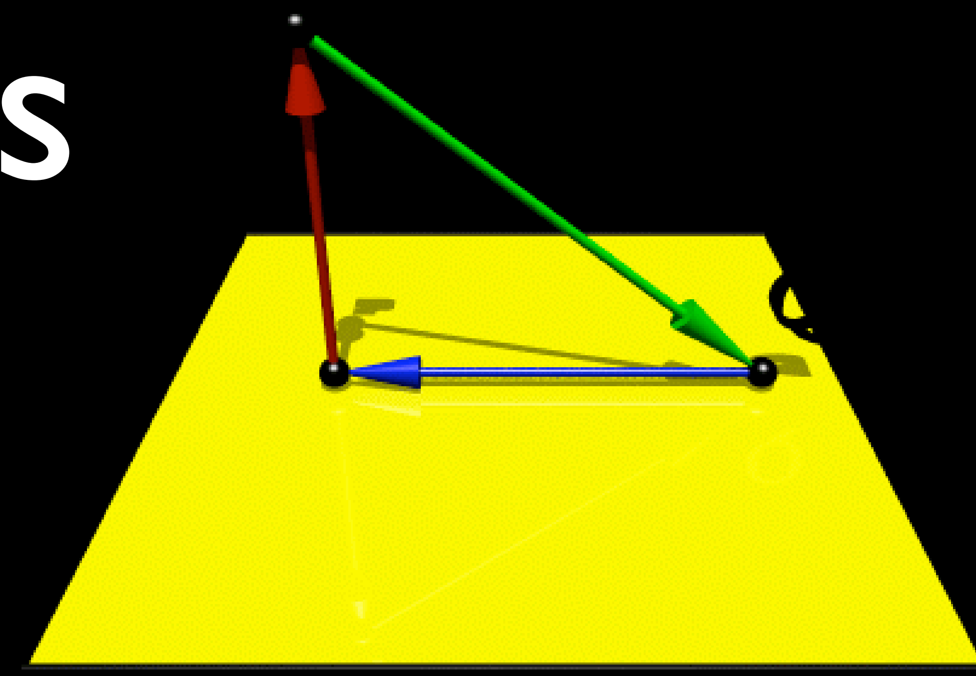


Project Q-P onto vxw

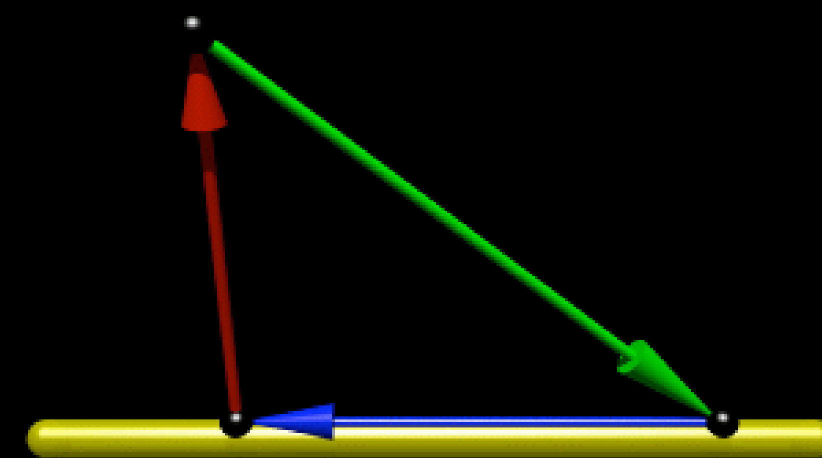
Distance Formulas



Distance Formulas

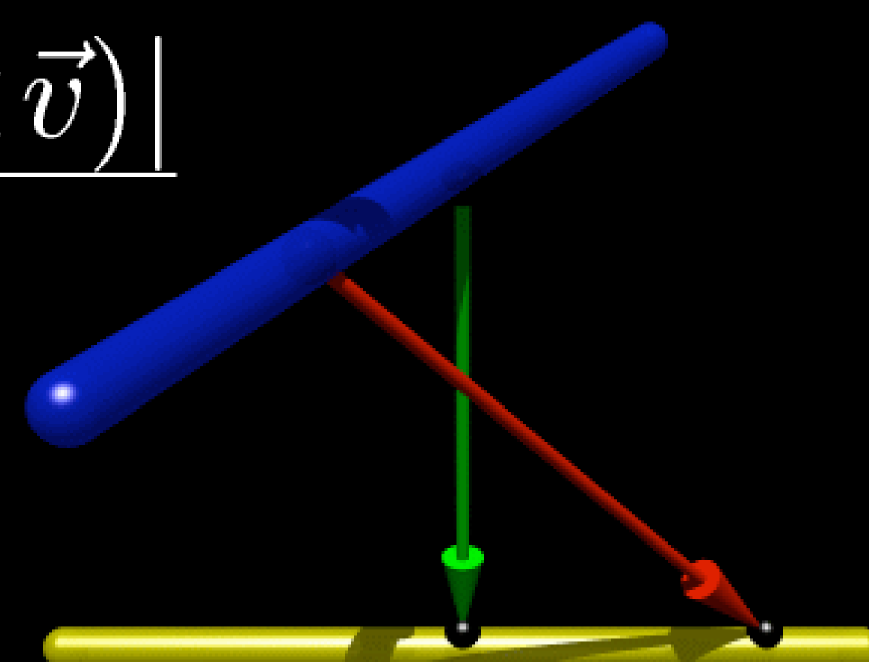


$$d(P,\Sigma) = \frac{|(\vec{PQ})\cdot\vec{n}|}{|\vec{n}|}$$

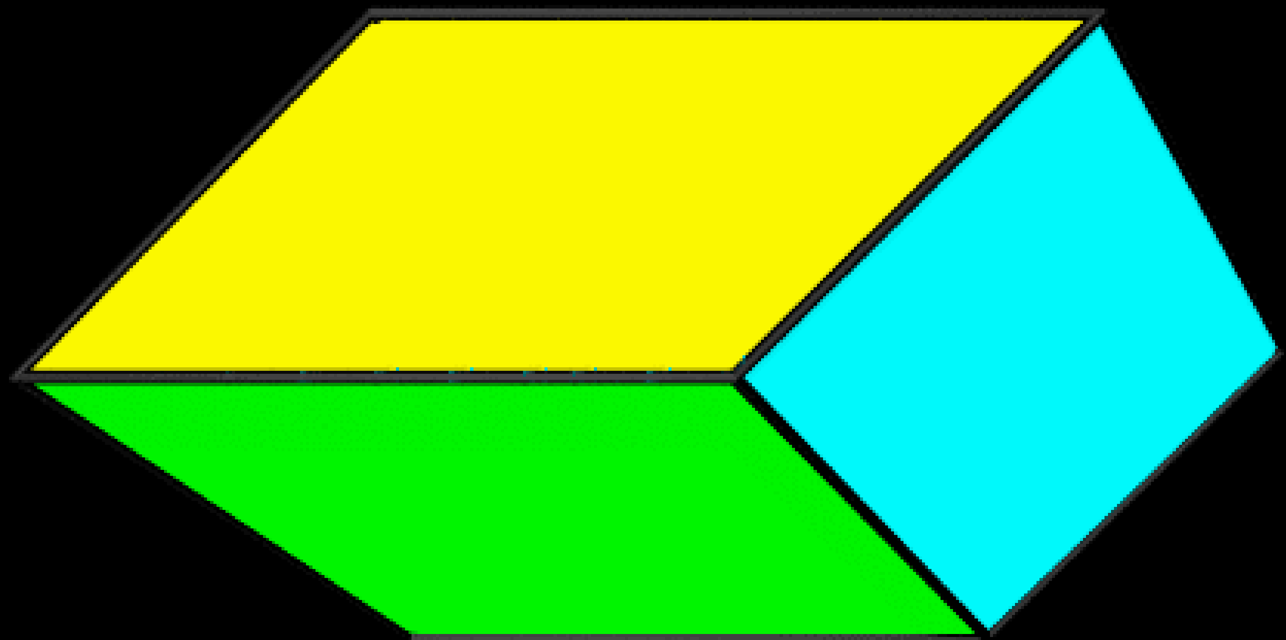


$$d(P,L) = \frac{|(\vec{PQ})\times\vec{u}|}{|\vec{u}|}$$

$$d(L,M) = \frac{|(\vec{PQ})\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$

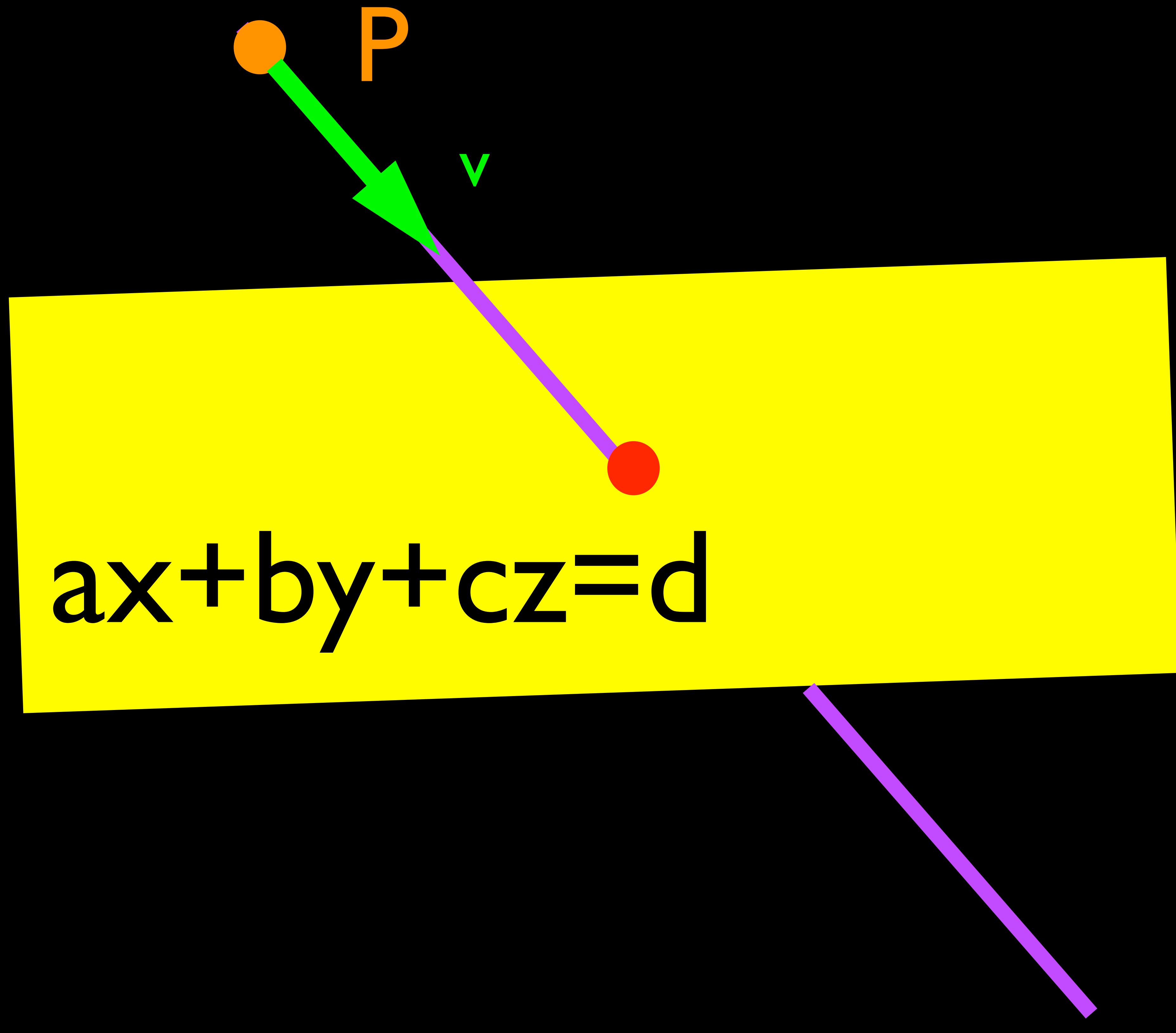


Parallelepiped

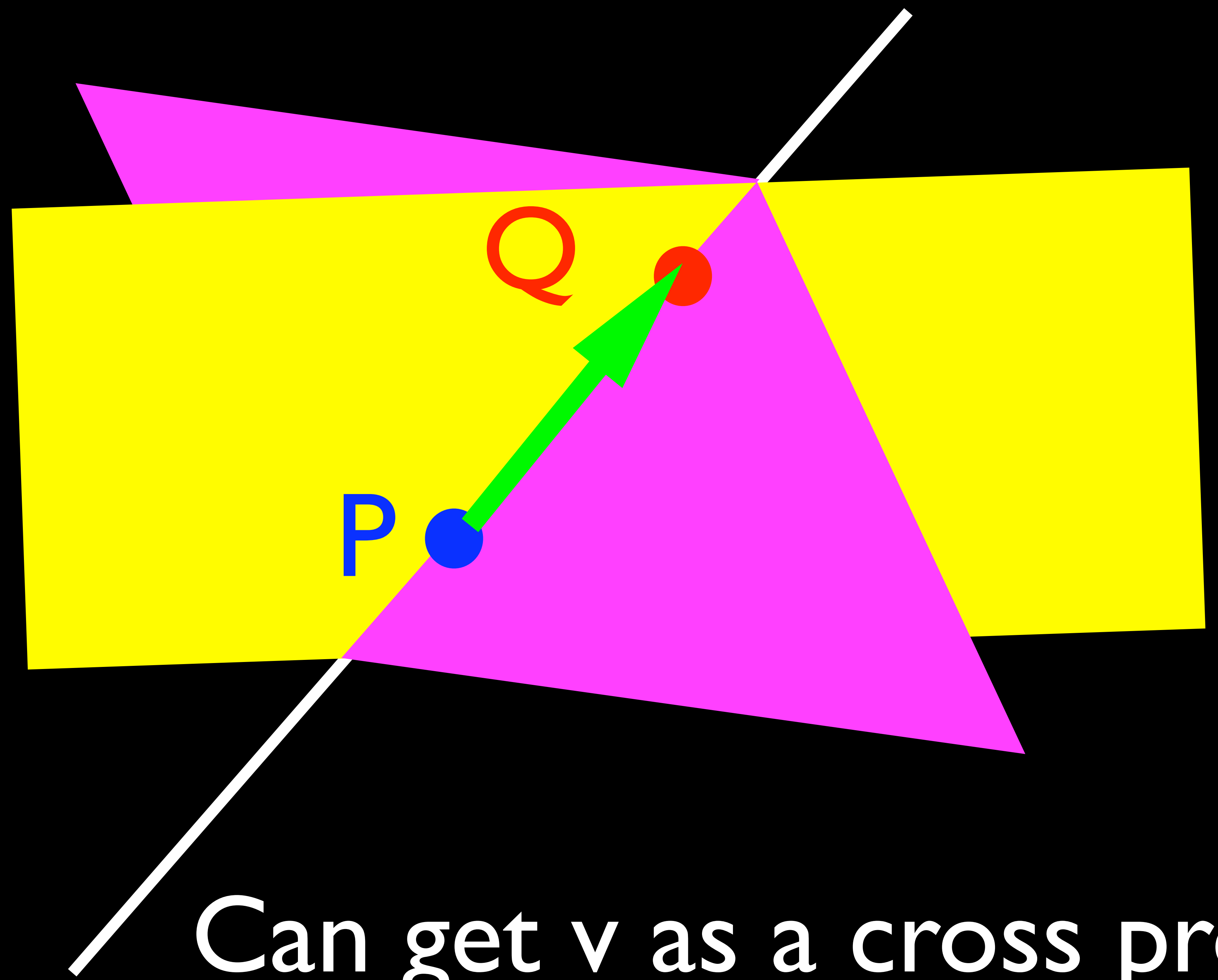


$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| \quad = \quad \text{Volume}$$

Plane and Line



Plane-Plane



Can get v as a cross product.

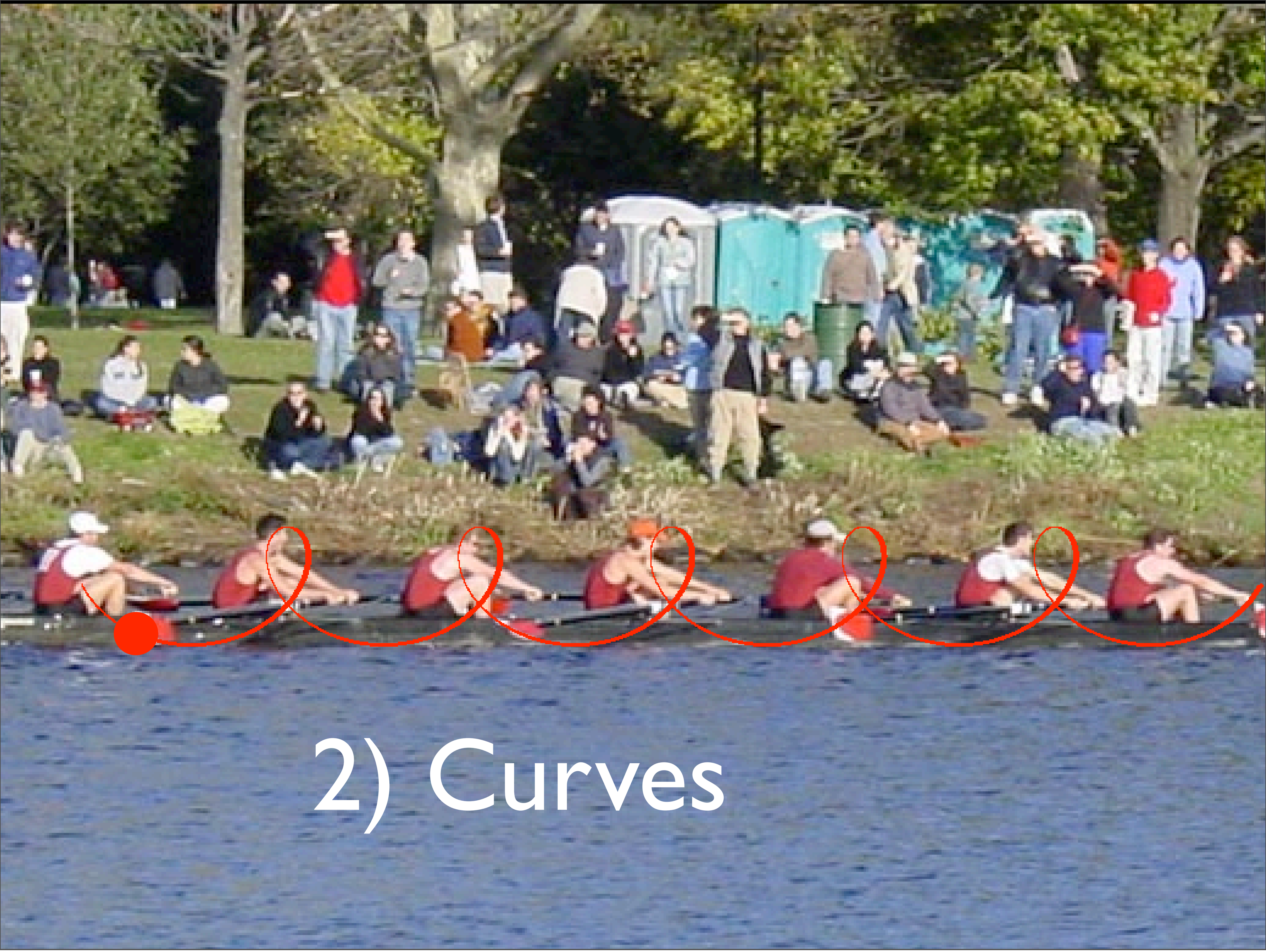
One Mars year = 586 days

Spirit, Dec 2005

Cydonia pyramide
Problem 3

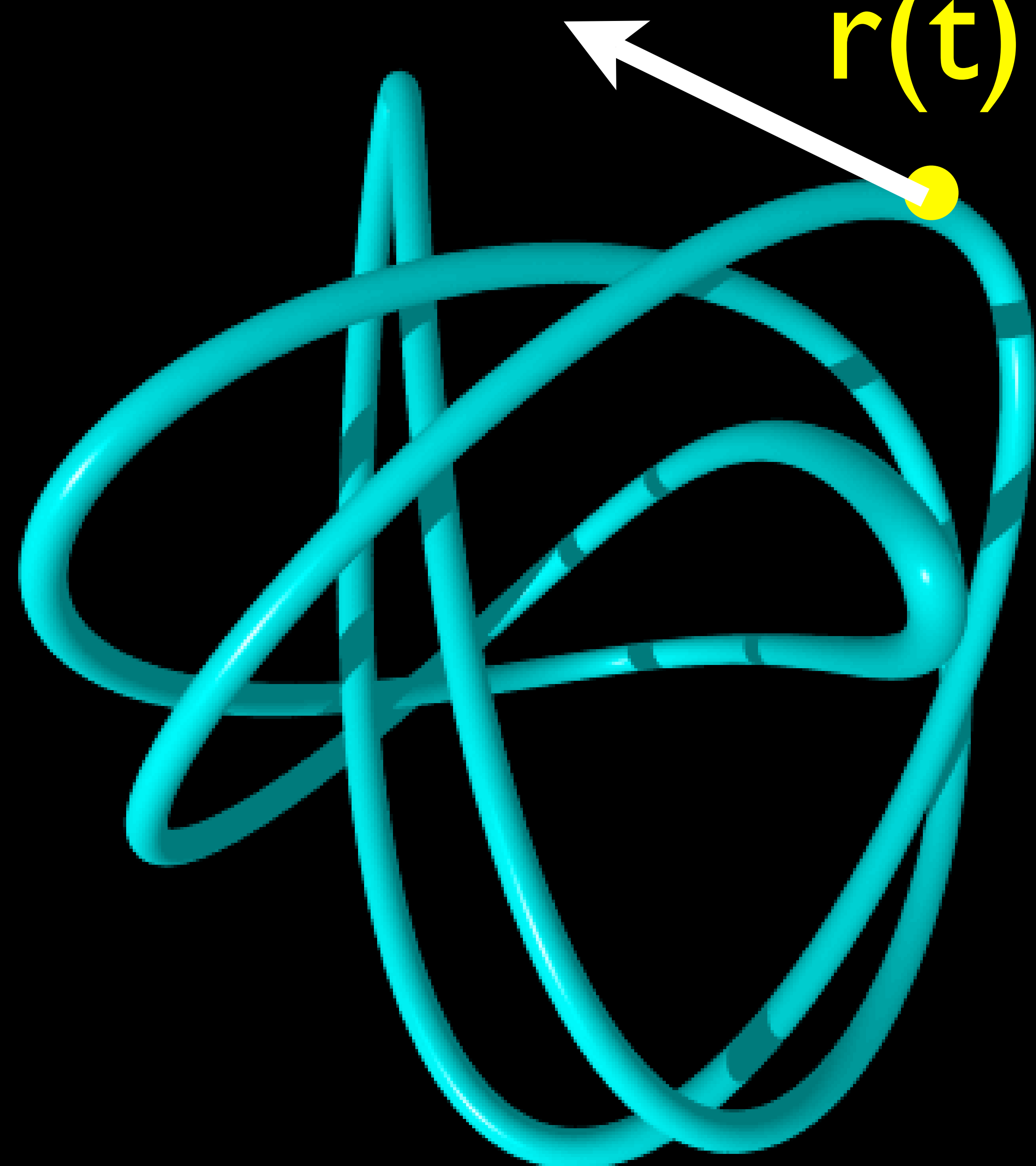
Find the distance from the tip of the cydonia pyramide on Mars with coordinates (1,-1,3) to
the surface modeled as the plane $x+2y+2z=1$.

2) Curves


2) Curves

Curves

Parametric Curves

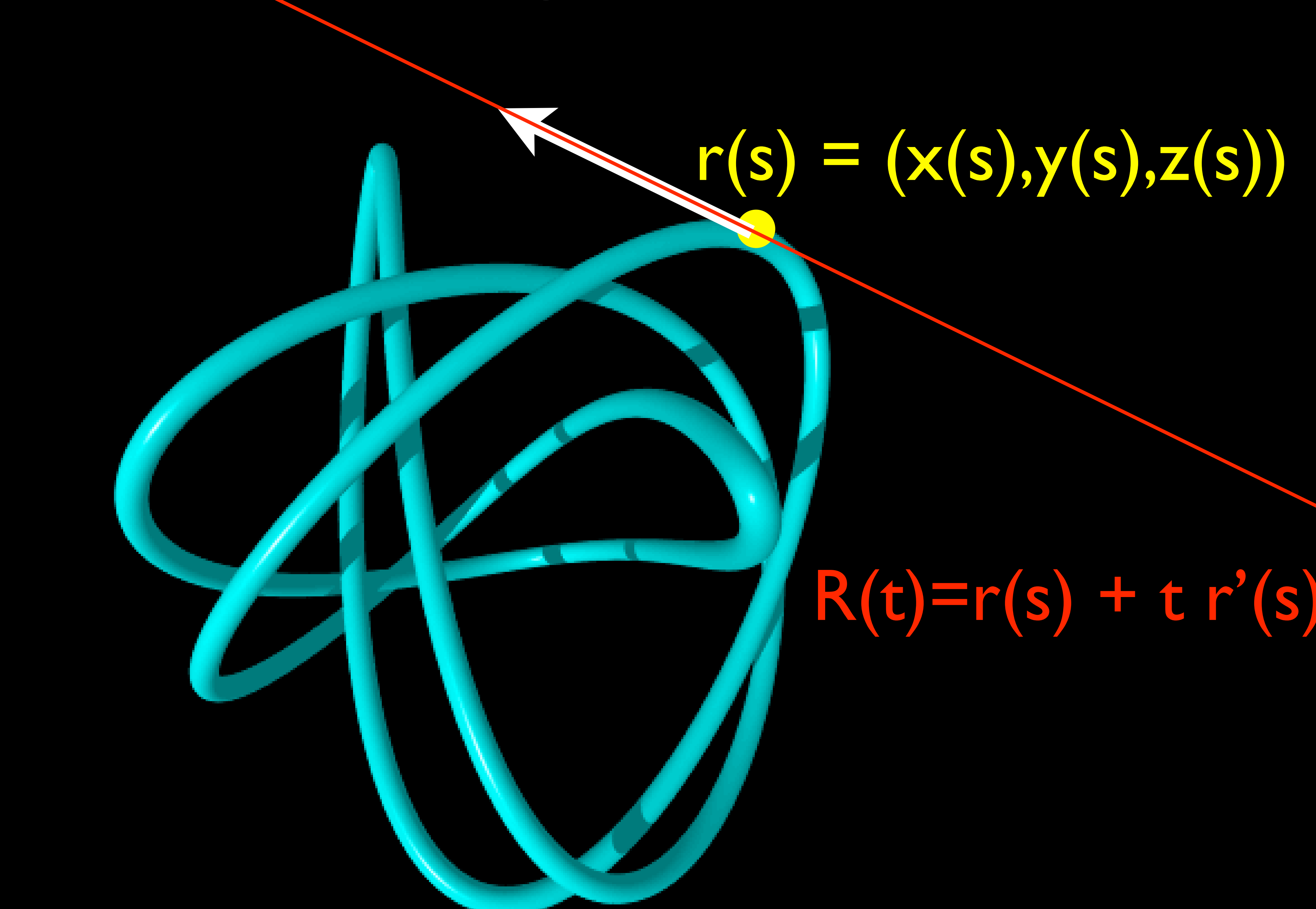


$r(t) = \langle x(t),y(t),z(t)\rangle$

Velocity: vector

Speed: length of velocity vector

Tangent Line



Integratation

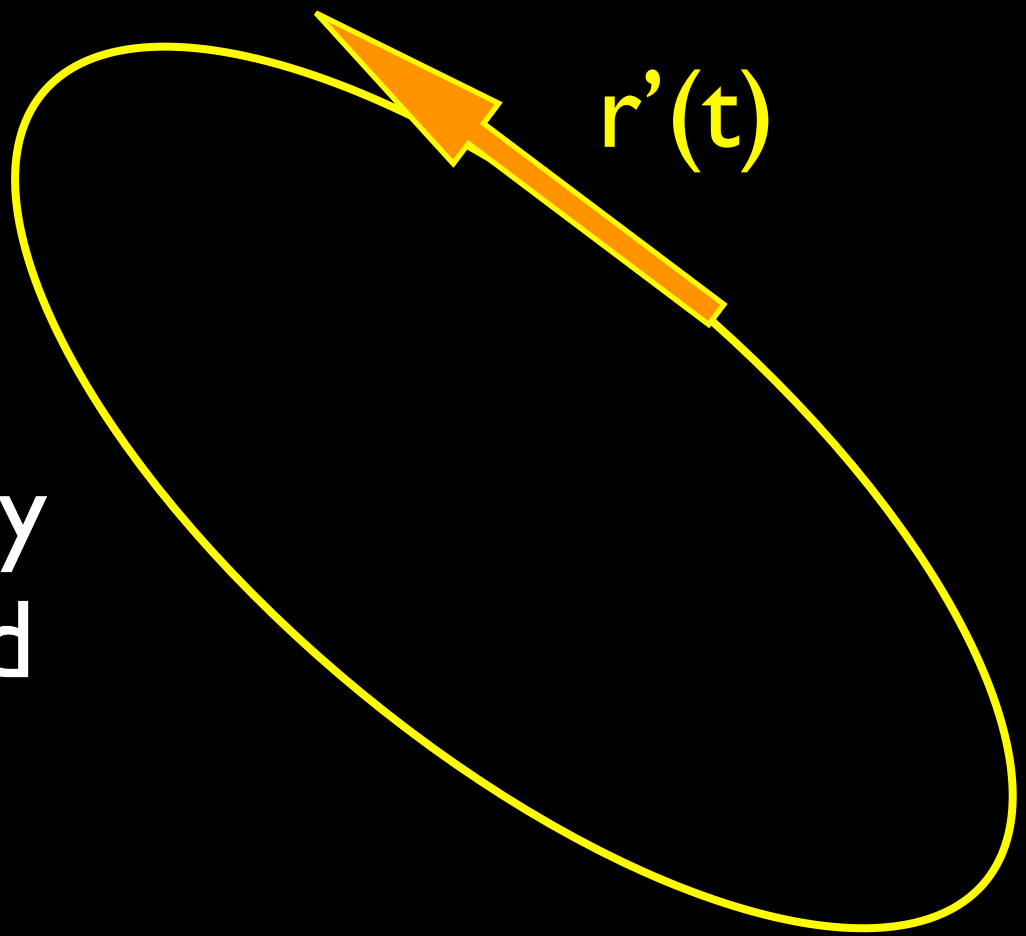
r ”(t) known at all times, r’(0) known, r(0) known, then r(t)= r(0)+r’(0) t + r”(t) t /2 is known

Even MS research is aware of this principle and built a prototype which knows where you are:

$$\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$$

$$\vec{v}(t) = \int_0^t \vec{r}''(s)\, ds, \vec{R}(t) = \int_0^t \vec{v}(s)\, ds$$

Arc Length



r'(t) velocity
|r'(t)| speed

Integrate speed over parameter interval to obtain arc length.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

The Frisbie Problem

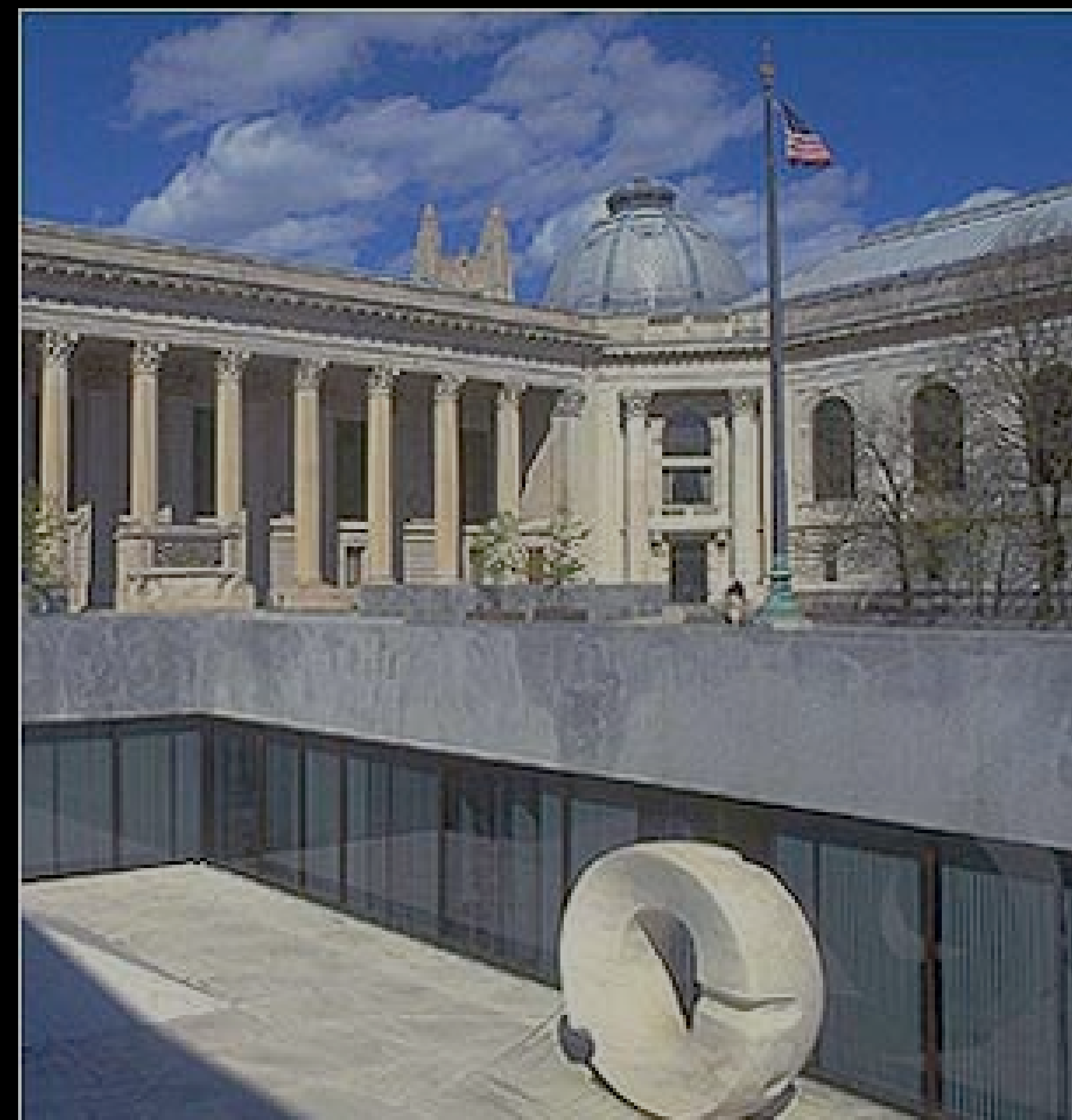
A frisbee flies on the following curve:

$$\vec{r}(t) = \langle \cos(e^t), e^t, \sin(e^t) \rangle$$

Find the length of the curve
from t=0 to t=1.

By the way...

Yale college has claimed to be the place where the frisbie was invented. The school has argued that a Yale undergraduate named Elihu Frisbie grabbed a passing collection tray from the chapel and flung it out into the campus, thereby becoming the inventor of the Frisbie and winning glory for Yale.



But ...

evenso Yale has “Lux and Veritas”
in their emblem, it
is no accident that
“lux” is above “veritas....

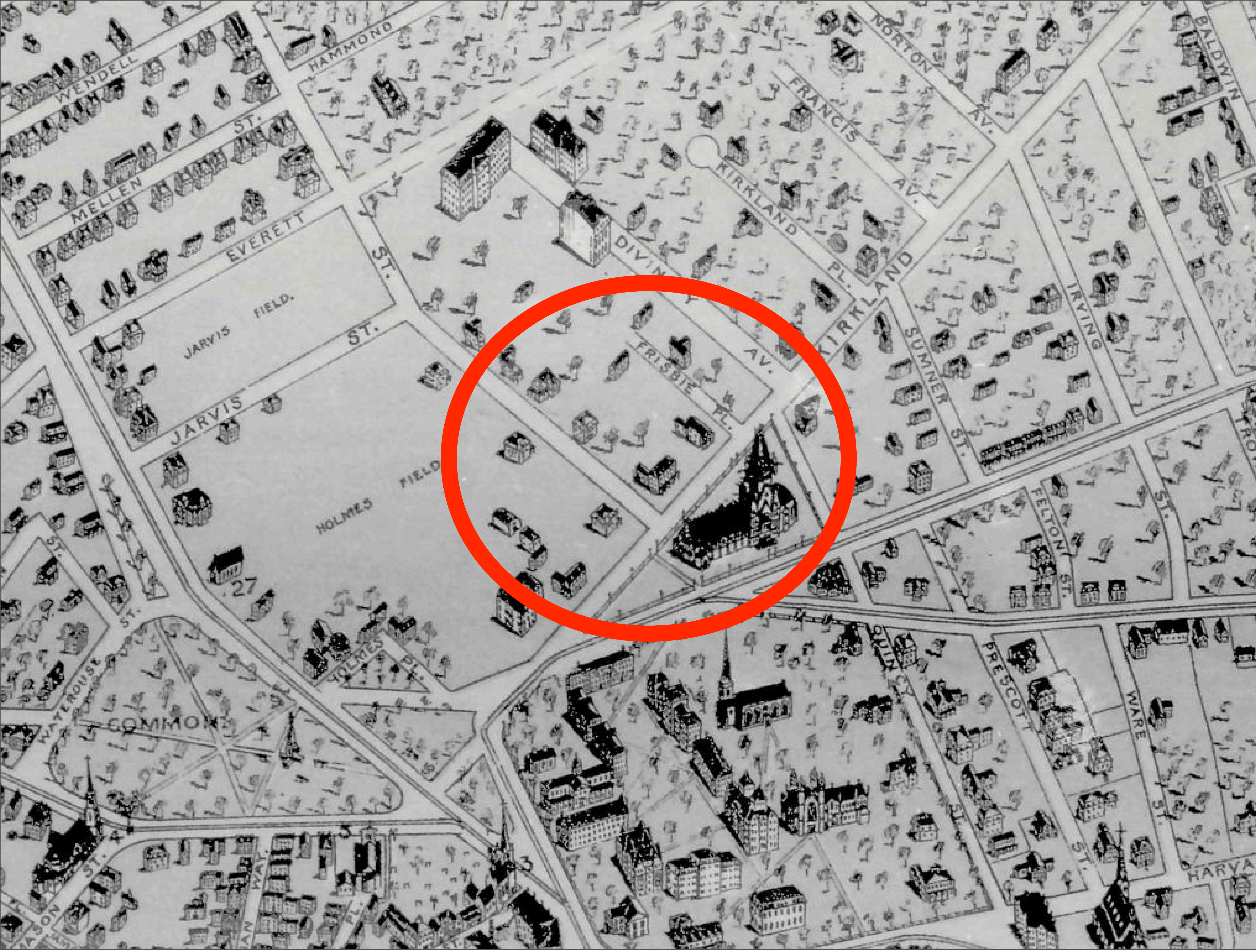
this story is not true. The frisbie was invented at Harvard

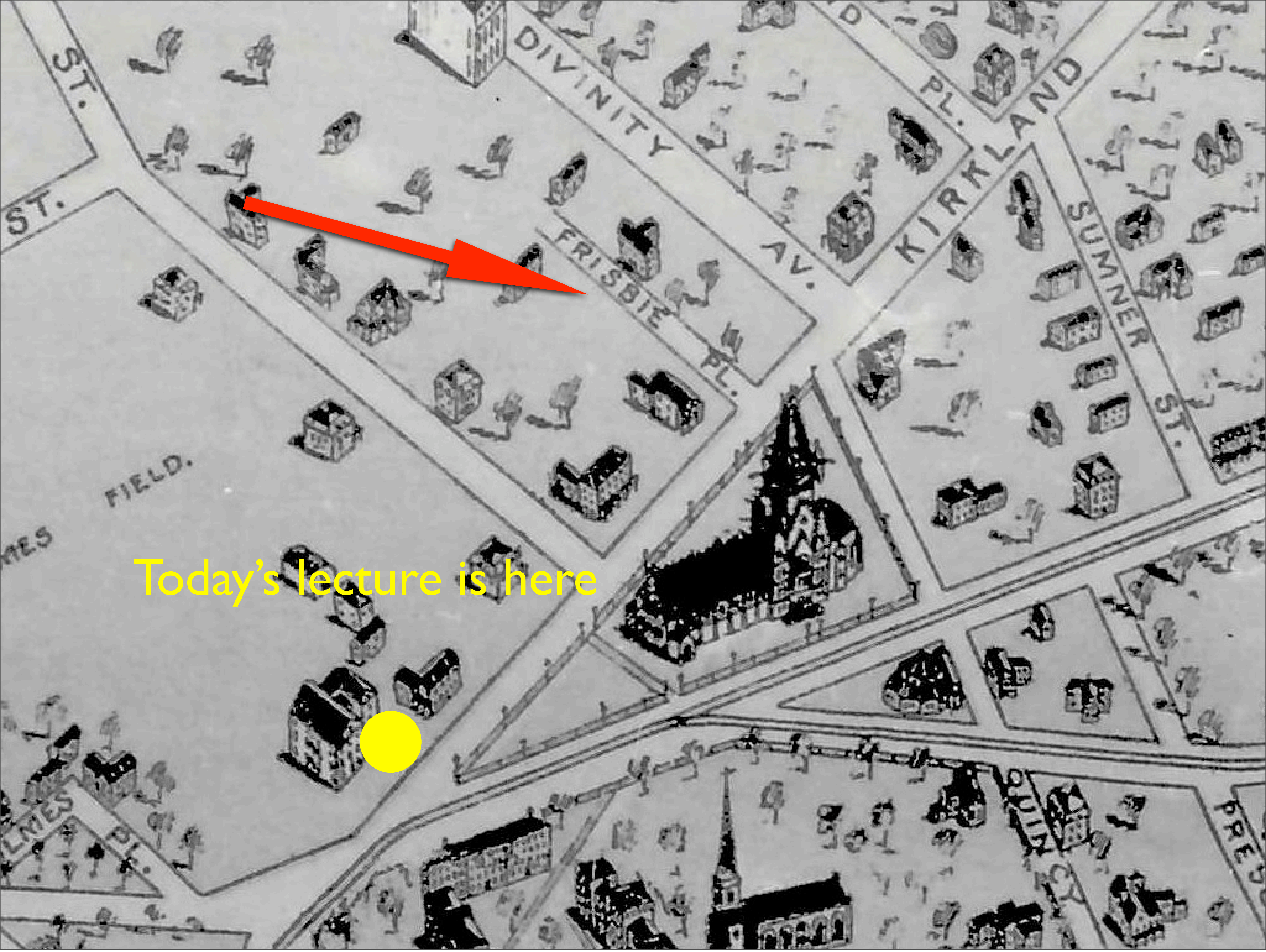
Lets go back in time

Cambridge 1775 ...

1854....
Keith
Station
R. Morse
Mrs. Holmes
Bapt. Church
Lawrence Scientific School
C.C. Foster
Ames
Dr. E. Howe
OXFORD
Prof. J. Farrar
Upham
DIVINITY
E.P. Whitman
Ware
KIRKLAND
DELTA
STREET
J. Sparks
Rice
H. Greenough
CAMBRIDG
Lieut Davis
Holworthy Hall
Stoughton Hall
Holden Hall
Hollis Hall
Harvard Hall
University Hall
Massachusetts Hall
HARVARD
Unitarian Church
STREET
Prof. C.C. Felton
A.H. Parker
Buckingham

Cambridge 1877 ...


HAMMOND
NORTON AV.
BALDWIN
WENDELL
ST.
MELLEN
EVERETT
FRANCIS AV.
KIRKLAND PL.
KIRKLAND
JARVIS FIELD.
JARVIS ST.
HOLMES FIELD
FRISBIE PL.
SUMNER ST.
IRVING
FELTON ST.
PRESCOTT ST.
WARE ST.
WATERHOUSE ST.
COMMON
HOLMES PL.
27

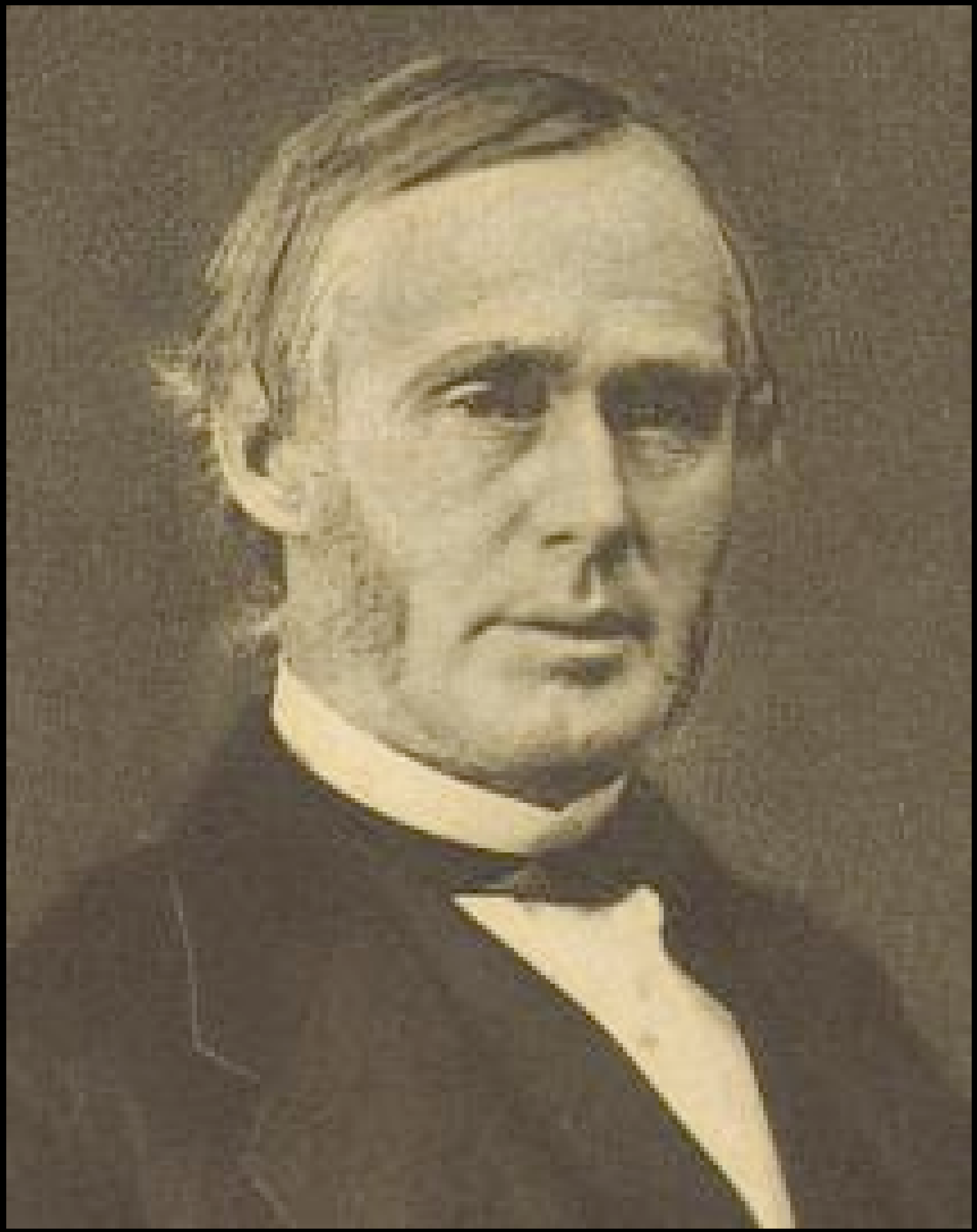

DIVINITY AV.
KIRKLAND
PL.
SUMNER ST.
FRISBIE PL.
ST.
ST.
FIELD.
Today's lecture is here

The fact is:

From 1871 to 1958, the Frisbie Baking company made pies that were sold to many New England colleges. Hungry Harvard students soon discovered that the empty pie tins could be tossed and cought, providing endless hours of game and sport.

One of the students was

George Frisbie Hoar graduated from Harvard University in 1846. Frisbie was often teased because his name could be found on every pie.

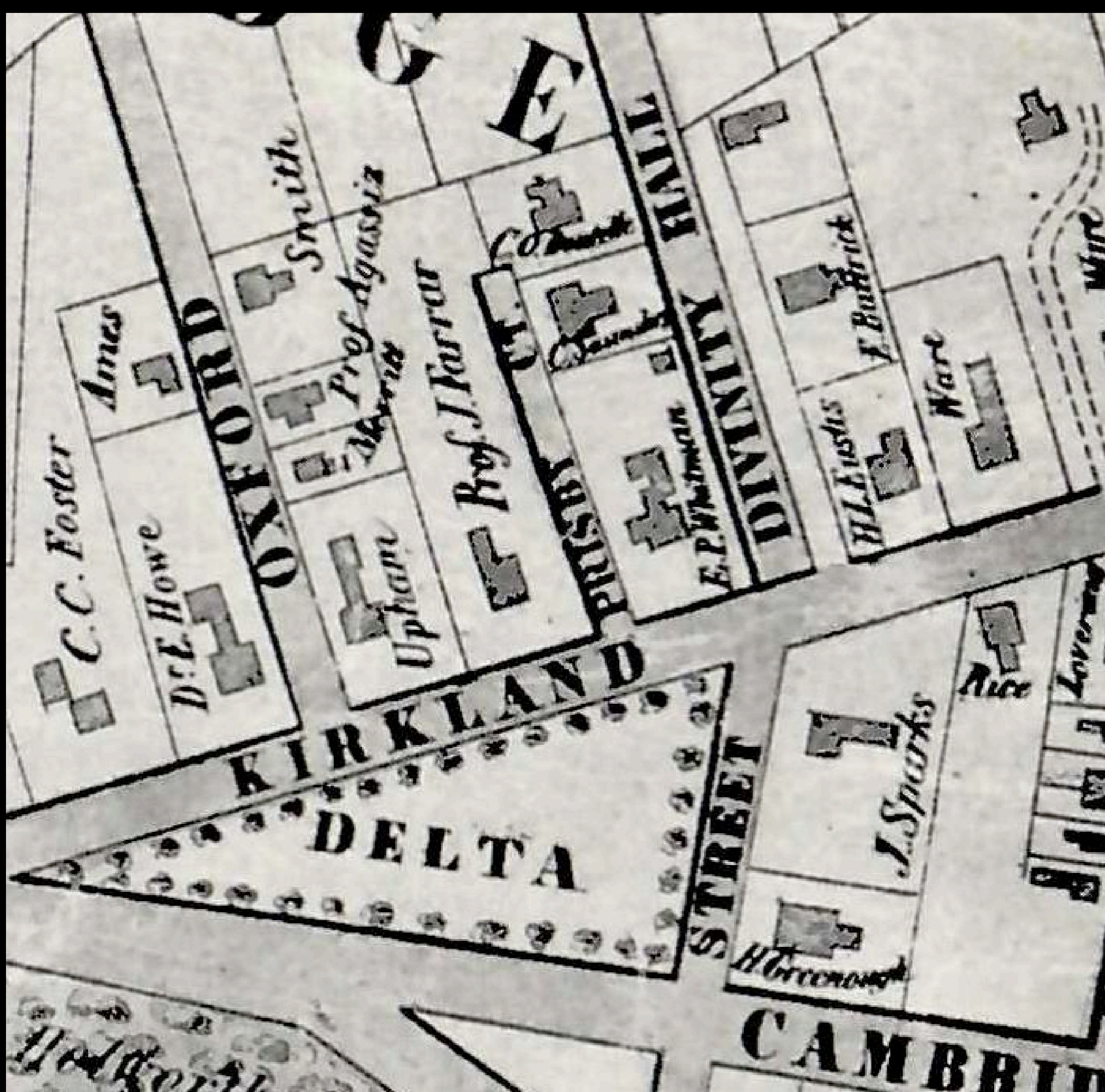


One beautiful march day as today,
at the exactly same spot as you sit now, a plate was thrown at him with the words: Frisbie, catch the Frisbie!

The frisbie was invented

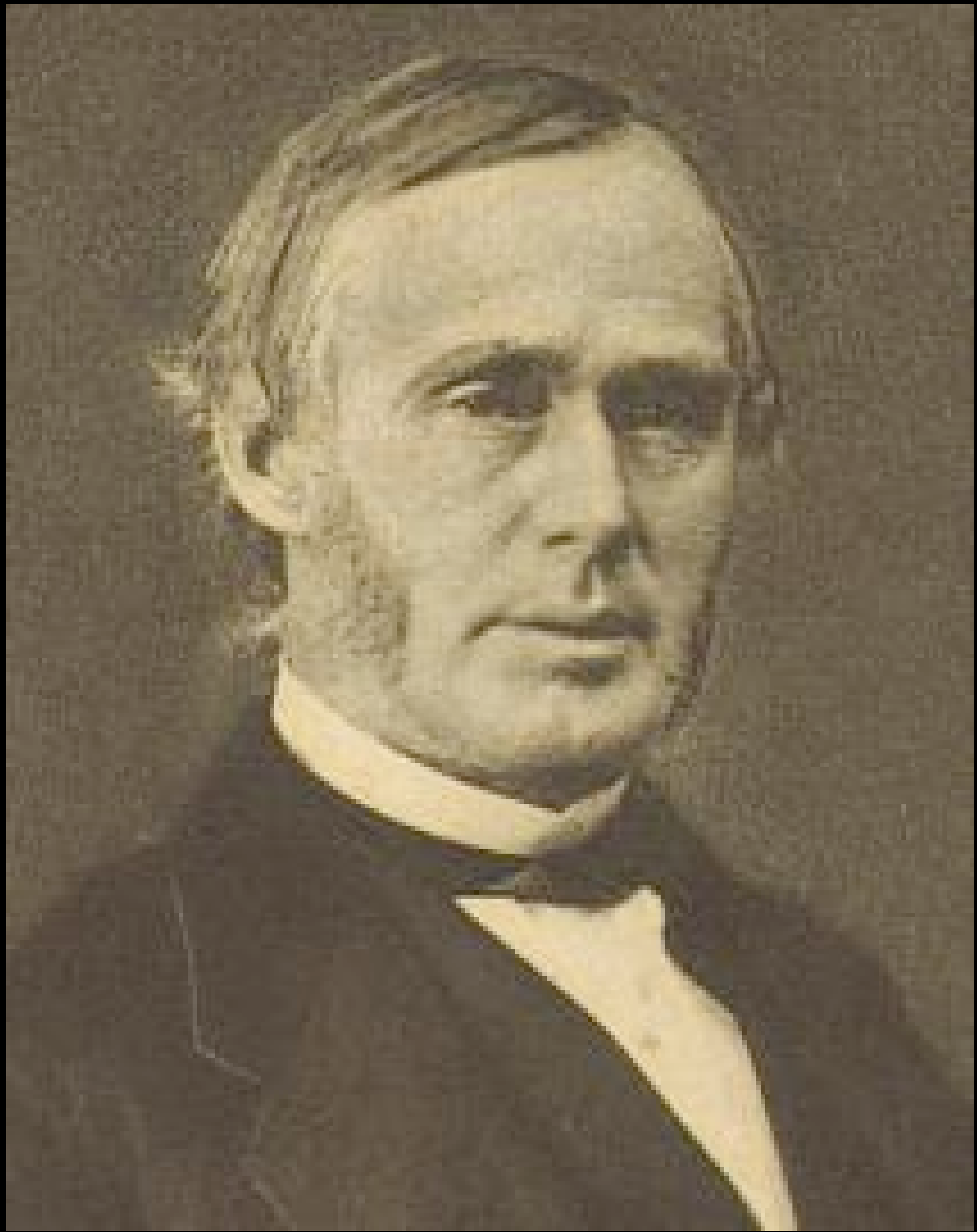
Note the transition from PRISBY to FRISBIE

1854

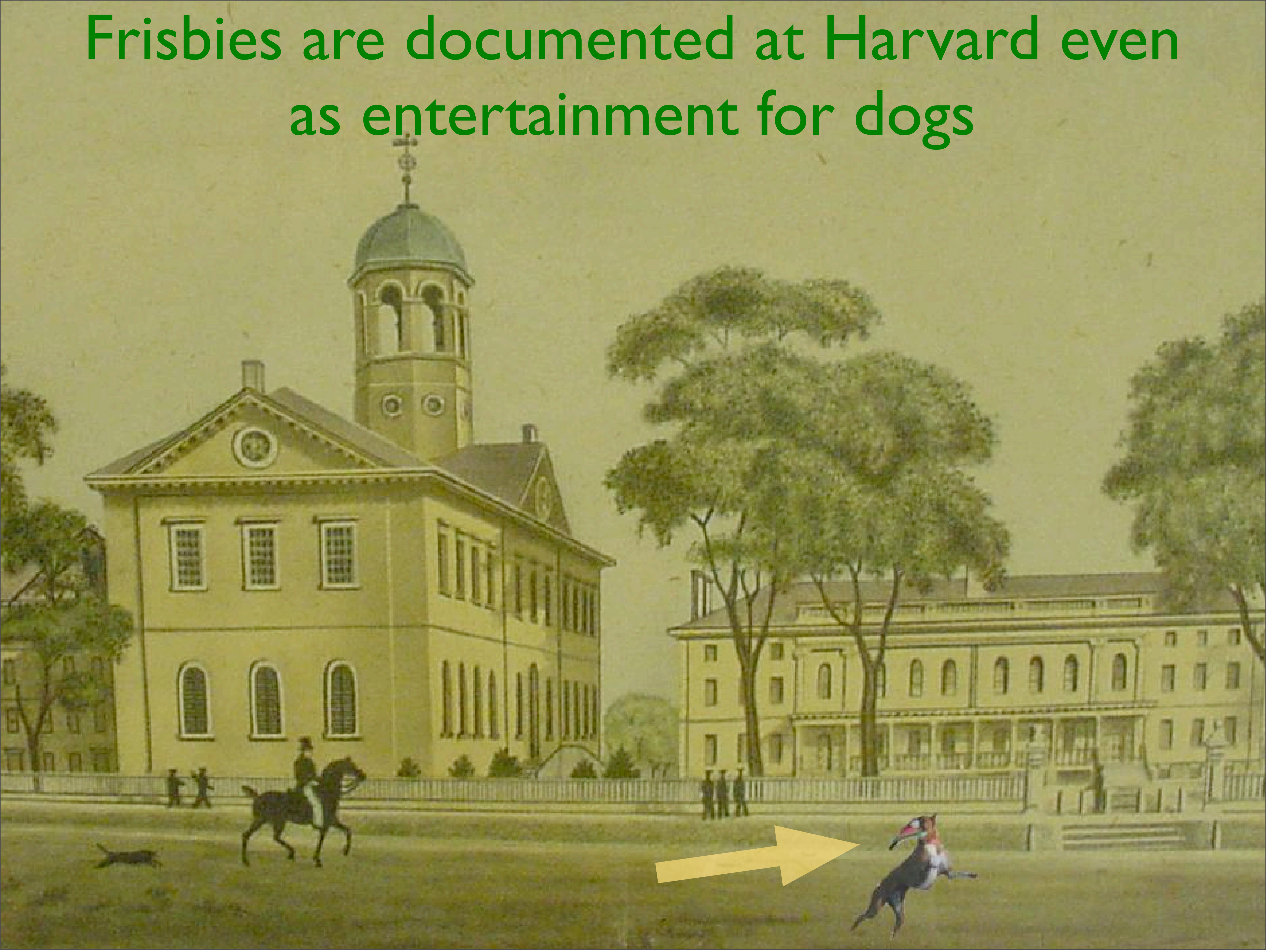


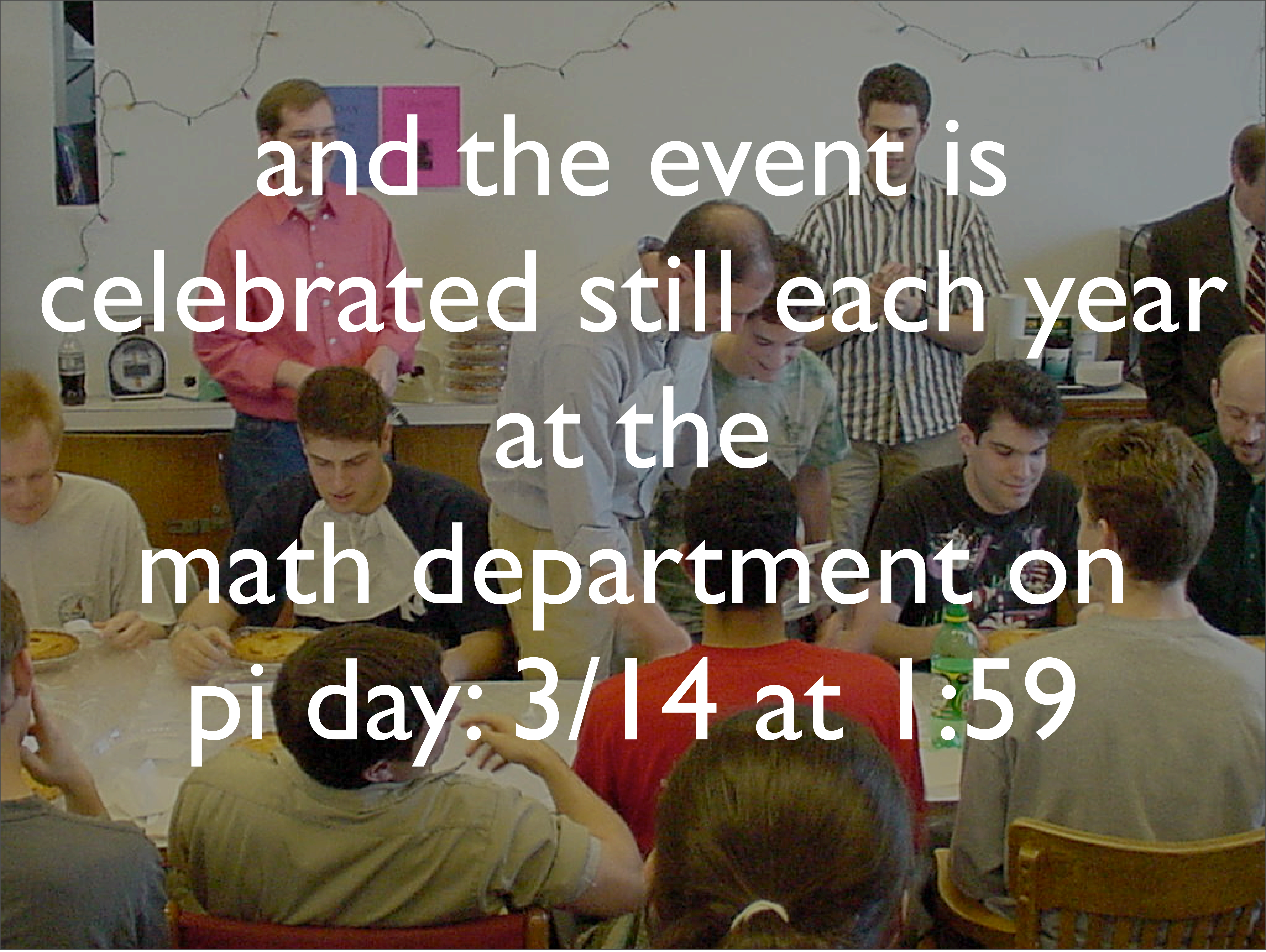
1877

George Frisbie Hoar (1826-1904), studied later law at Harvard Law School and served on the Mass state senate and the United States House of Representatives.



Frisbies are documented at Harvard even as entertainment for dogs





and the event is celebrated still each year at the math department on pi day: 3/14 at 1:59

Back to the problem

$$\vec{r}(t) = \langle \cos(e^t), e^t, \sin(e^t) \rangle$$

Find the length of the curve
from t=0 to t=1.

Product Rules

$$\frac{d}{dt}\left(\vec{v}(t)\cdot\vec{w}(t)\right) = \vec{v}\,'(t)\cdot\vec{w}(t) + \vec{v}(t)\cdot\vec{w}\,'(t)$$

$$\frac{d}{dt}\left(\vec{v}(t)\times\vec{w}(t)\right) = \vec{v}\,'(t)\times\vec{w}(t) + \vec{v}(t)\times\vec{w}\,'(t)$$

Application:

$$\frac{d}{dt}\left(\vec{r}(t)\times\vec{r}\,'(t)\right) = 0$$

Angular momentum conservation

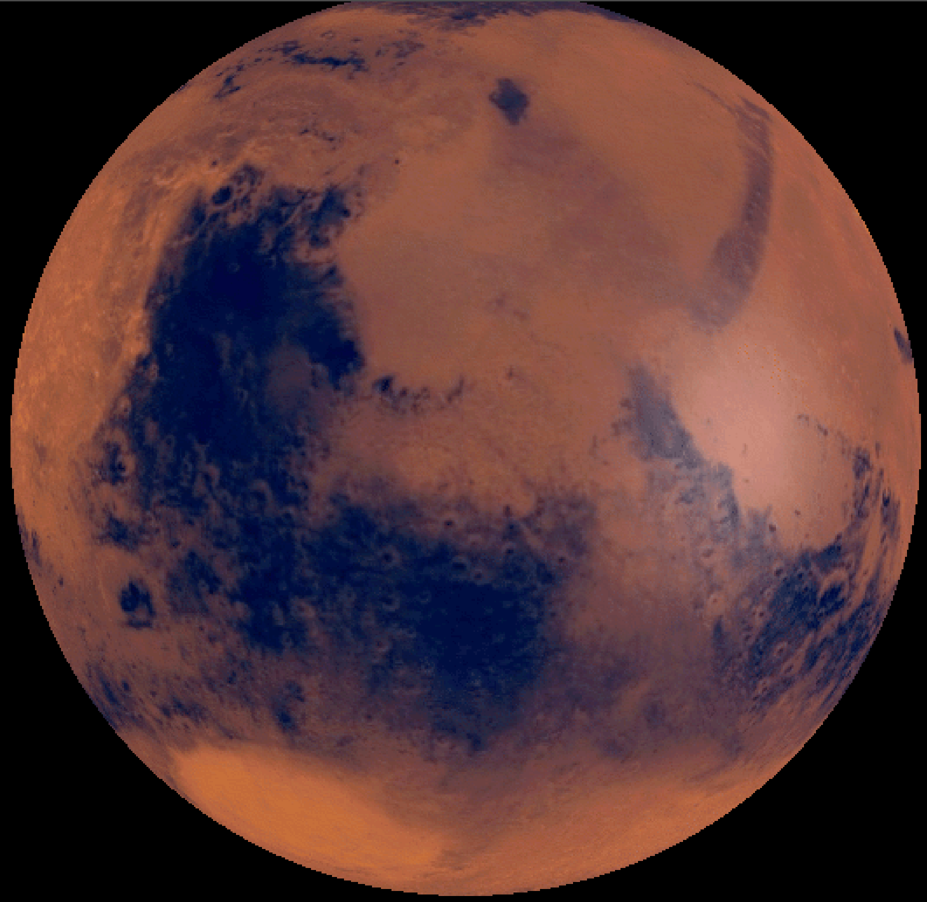
The Mars-bug riddle

Opportunity has photographed a "bunny"-shaped yellow object of about
4-5 centimeters diameter.

Curvature Problem

An mars bug flies along the path

$r(t) = (50\cos(t), 50\sin(t), 10)$

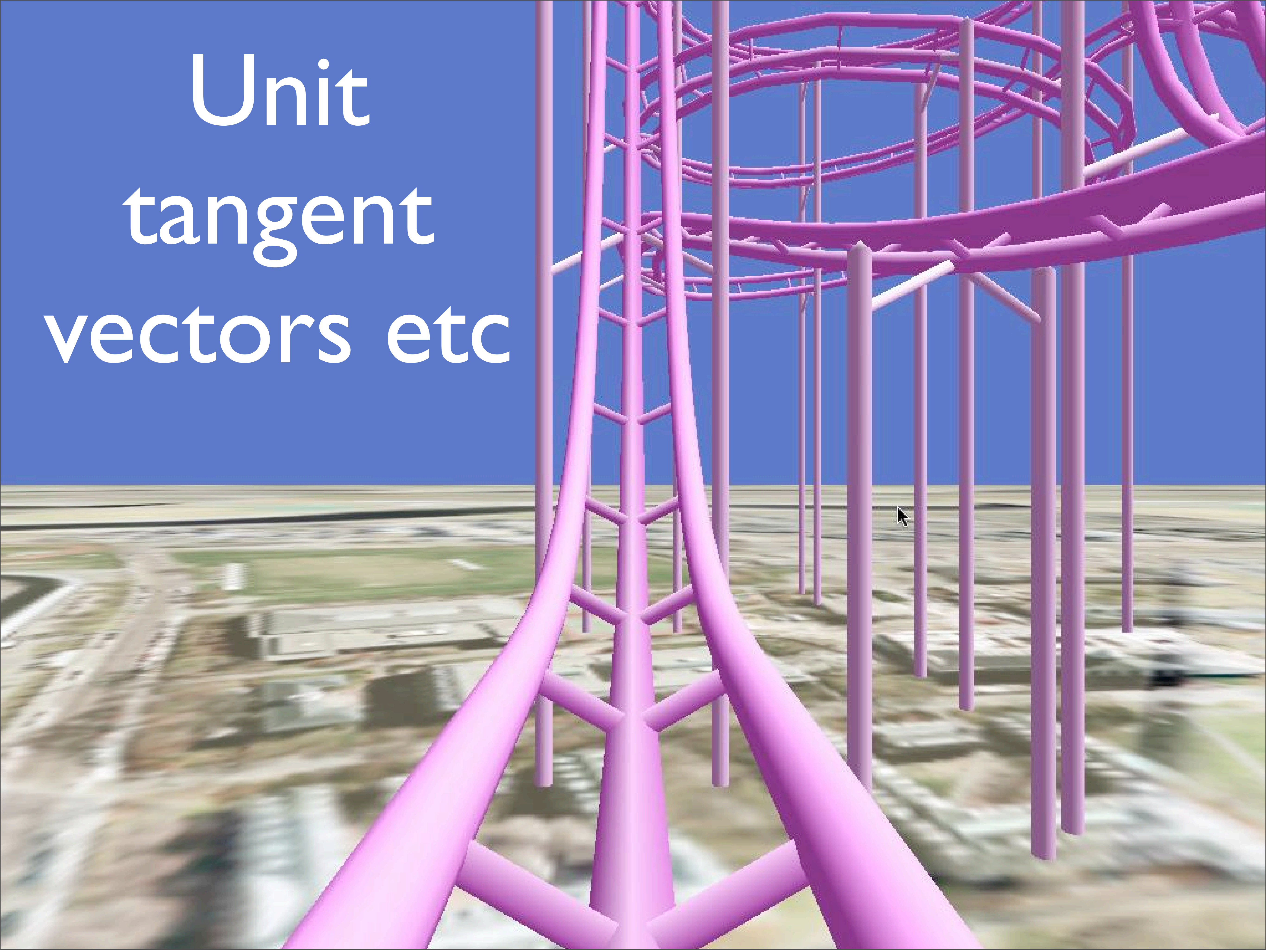


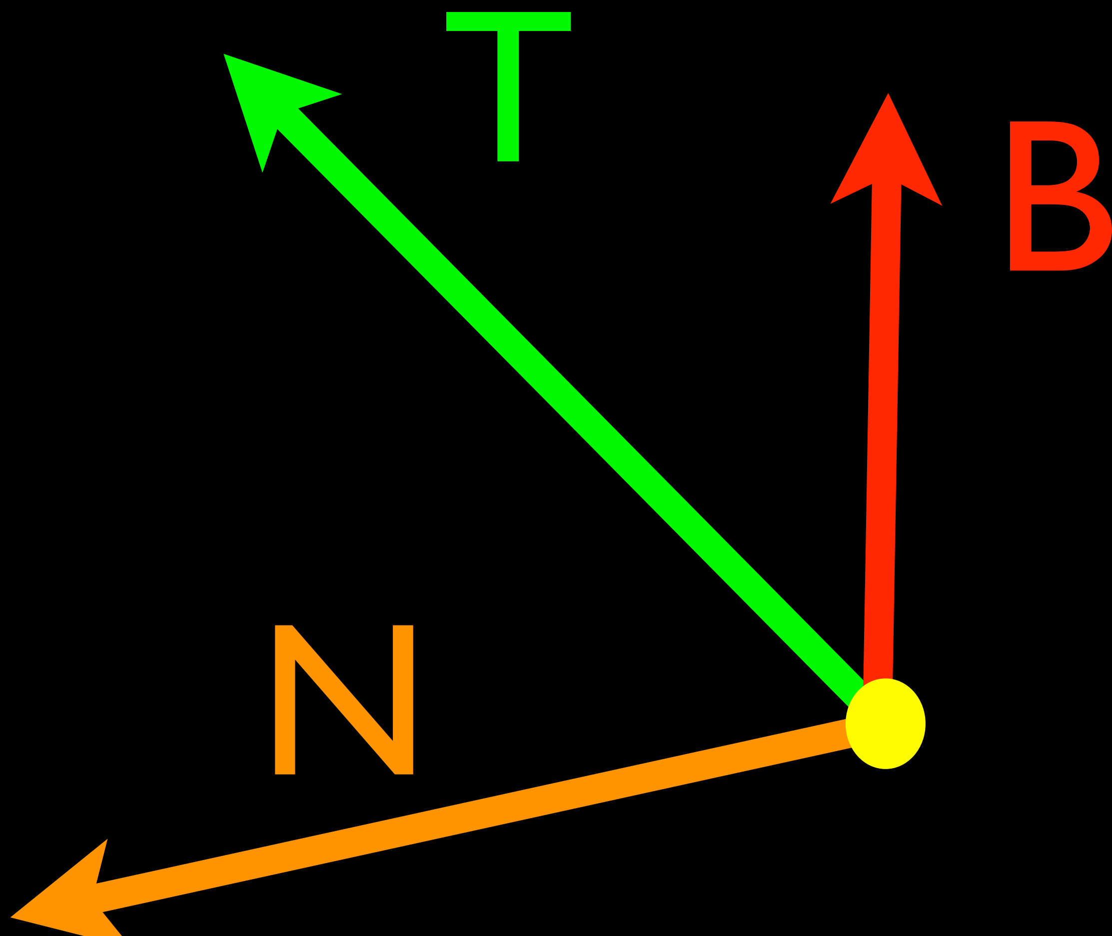
The mars rover travels along the path

$r(t) = (2\cos(t), 2\sin(t), 10)$

Which path has larger curvature?

Unit tangent vectors etc

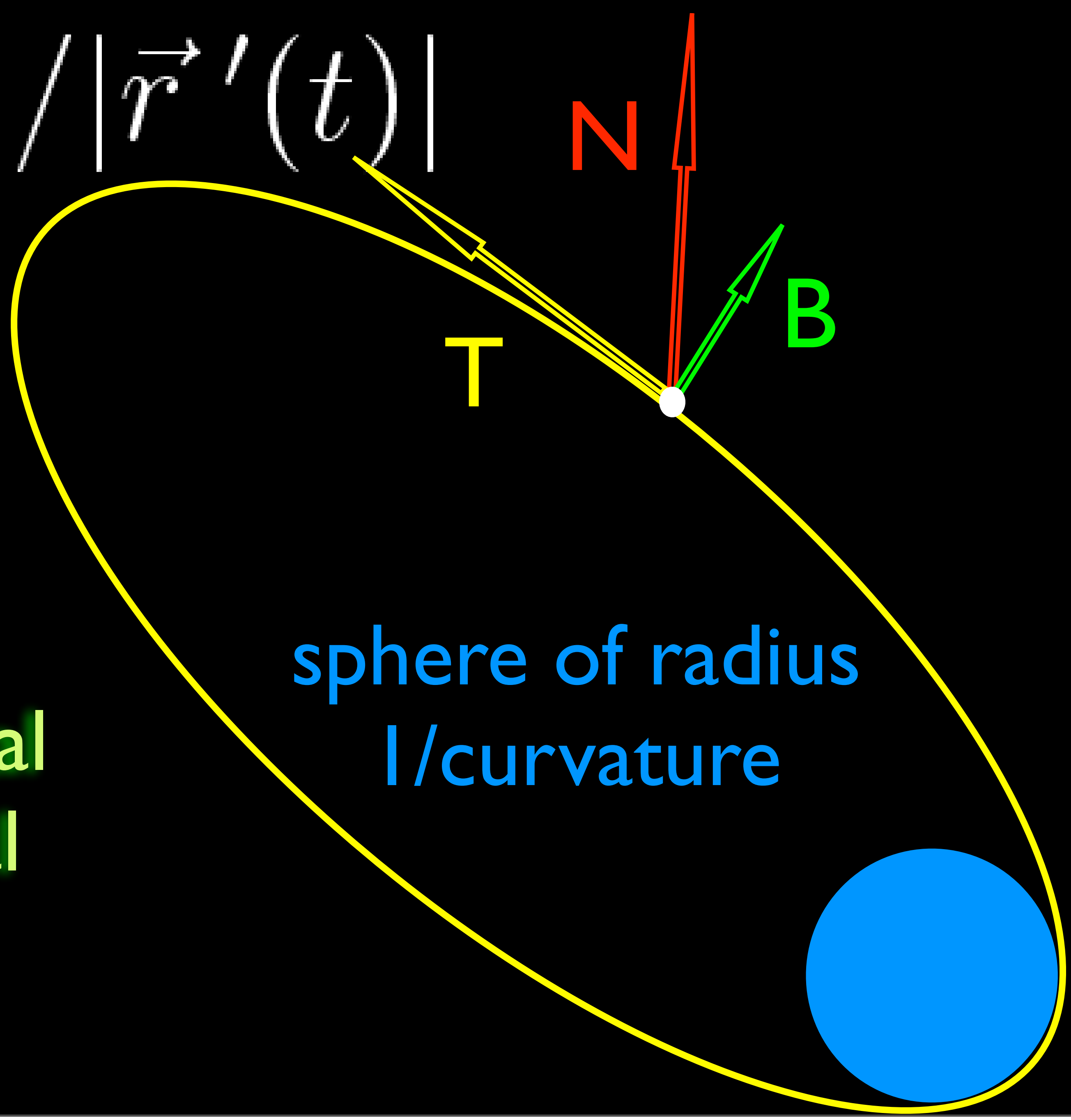



T
B
N


2 cd set
GOTHIC VAMPIRES
FROM
HELL
&
COVERED
IN

Formulas

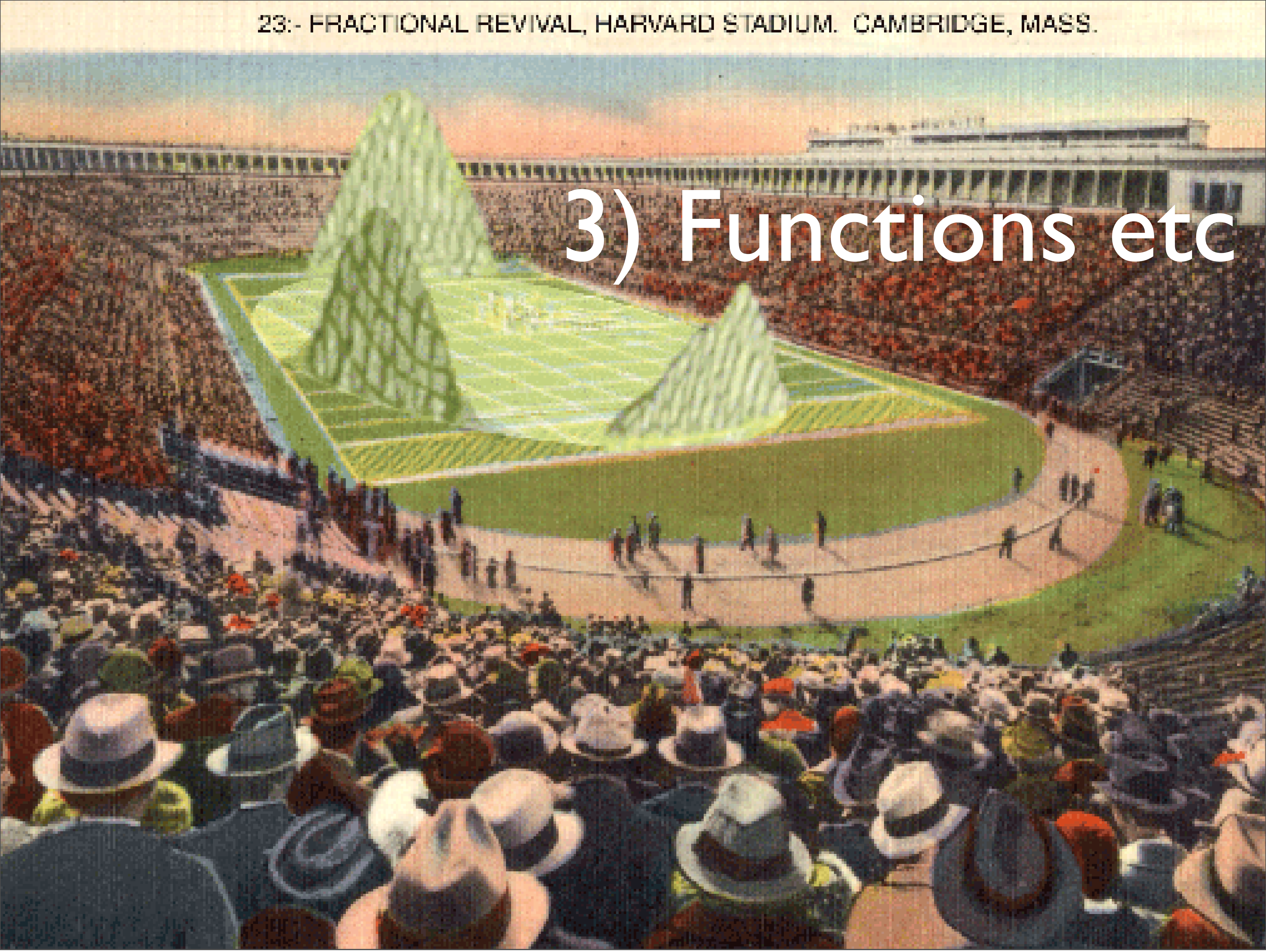
$$\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$$
$$\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$$
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$
$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)|$$
$$= \frac{|r\,'(t) \times r\,''(t)|}{|r\,'(t)|^3}$$



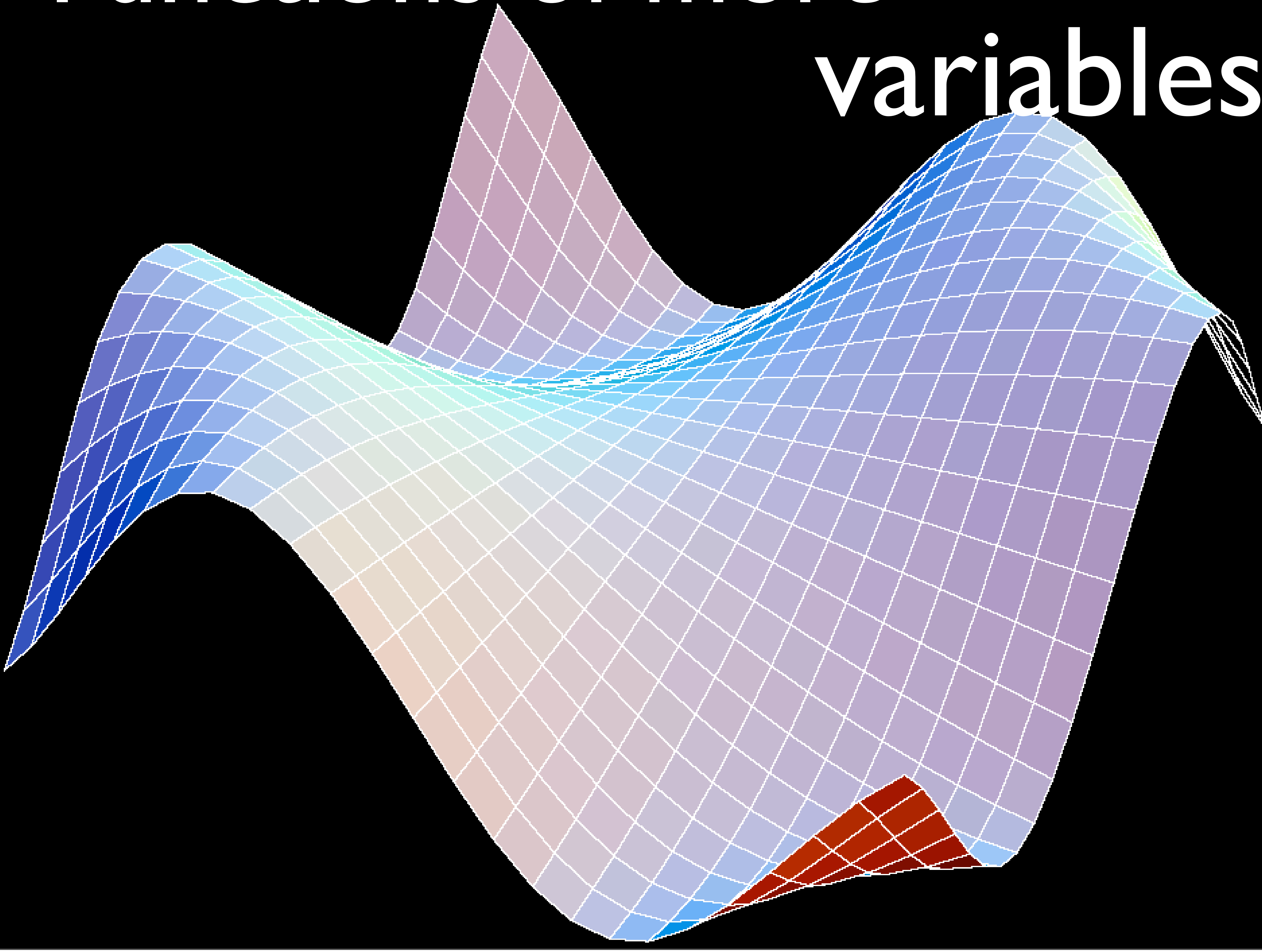
Unit tangent, unit normal and binormal vectors ore normal to each other

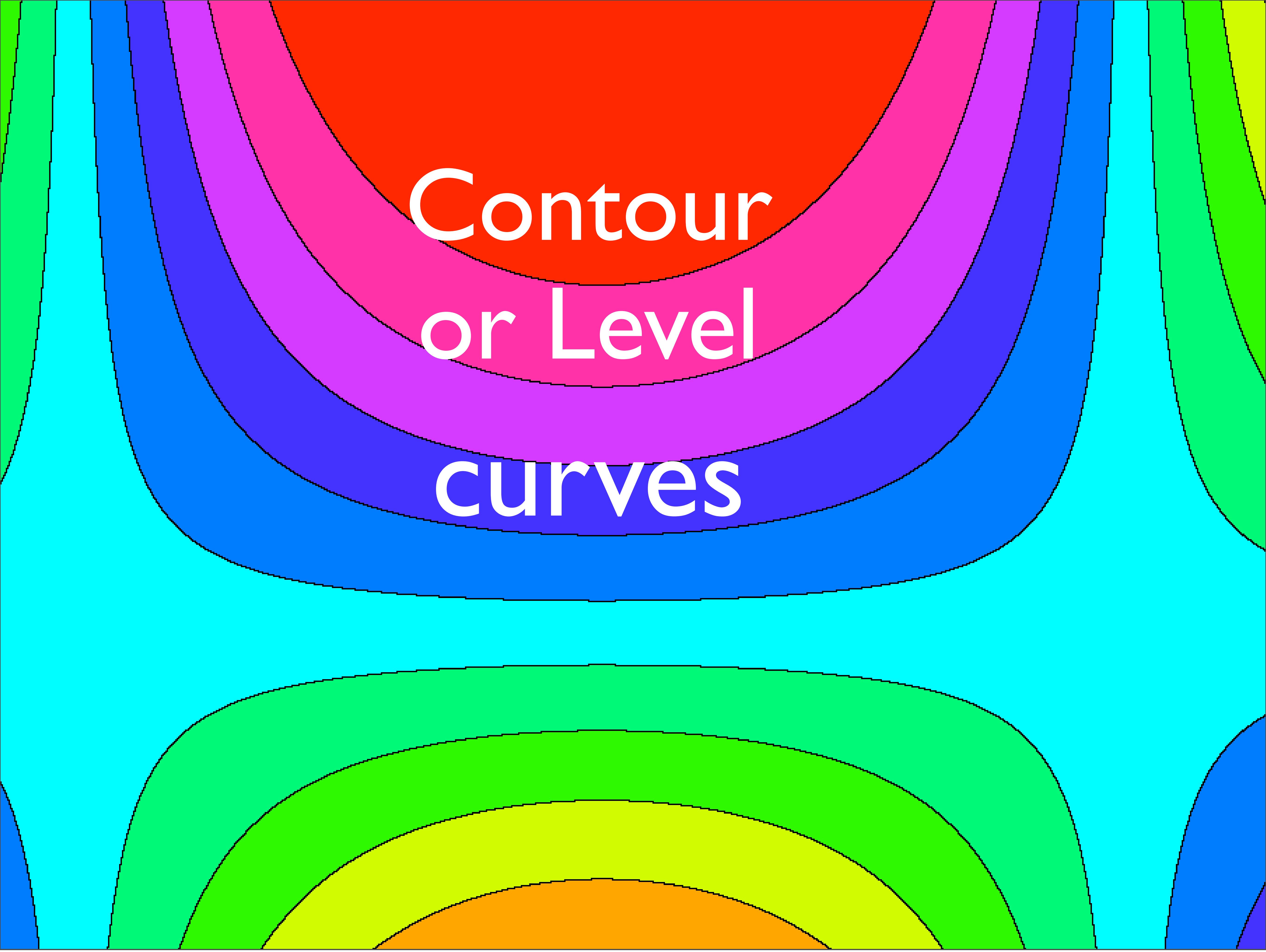
Thats all I have to say about that.

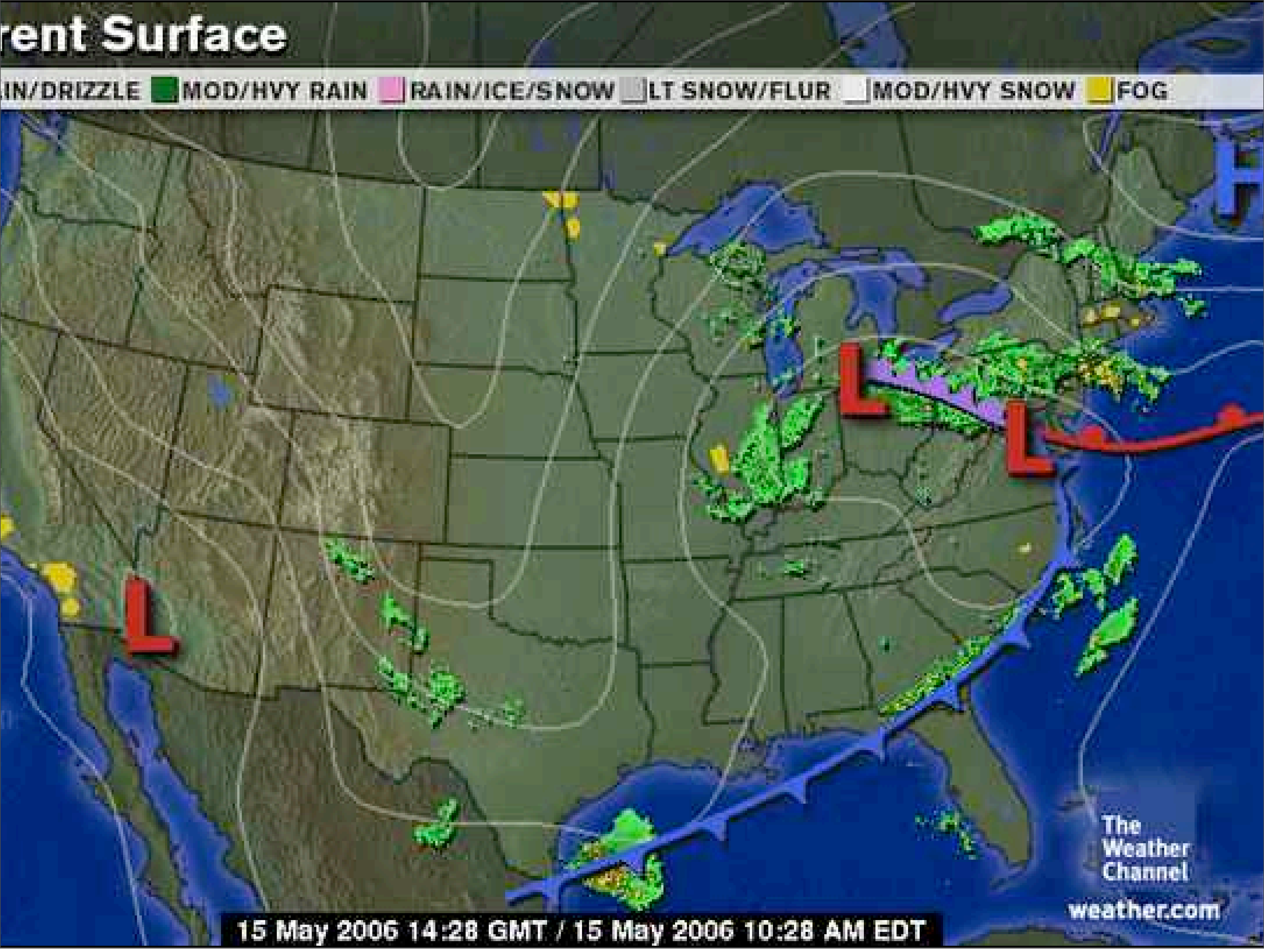
3) Functions etc



Functions of more variables




Contour
or Level
curves


rent Surface
IN/DRIZZLE
MOD/HVY RAIN
RAIN/ICE/SNOW
LT SNOW/FLUR
MOD/HVY SNOW
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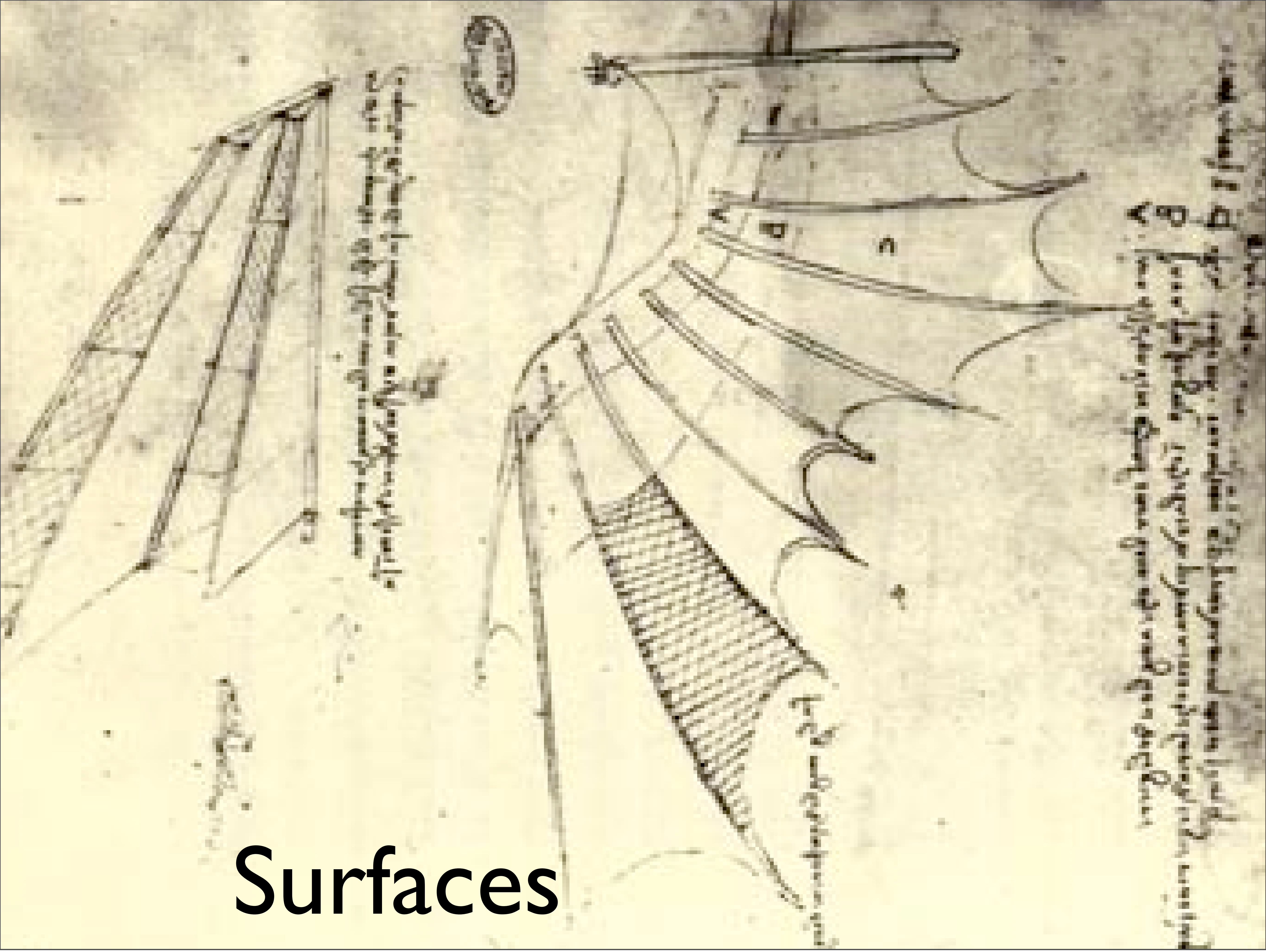
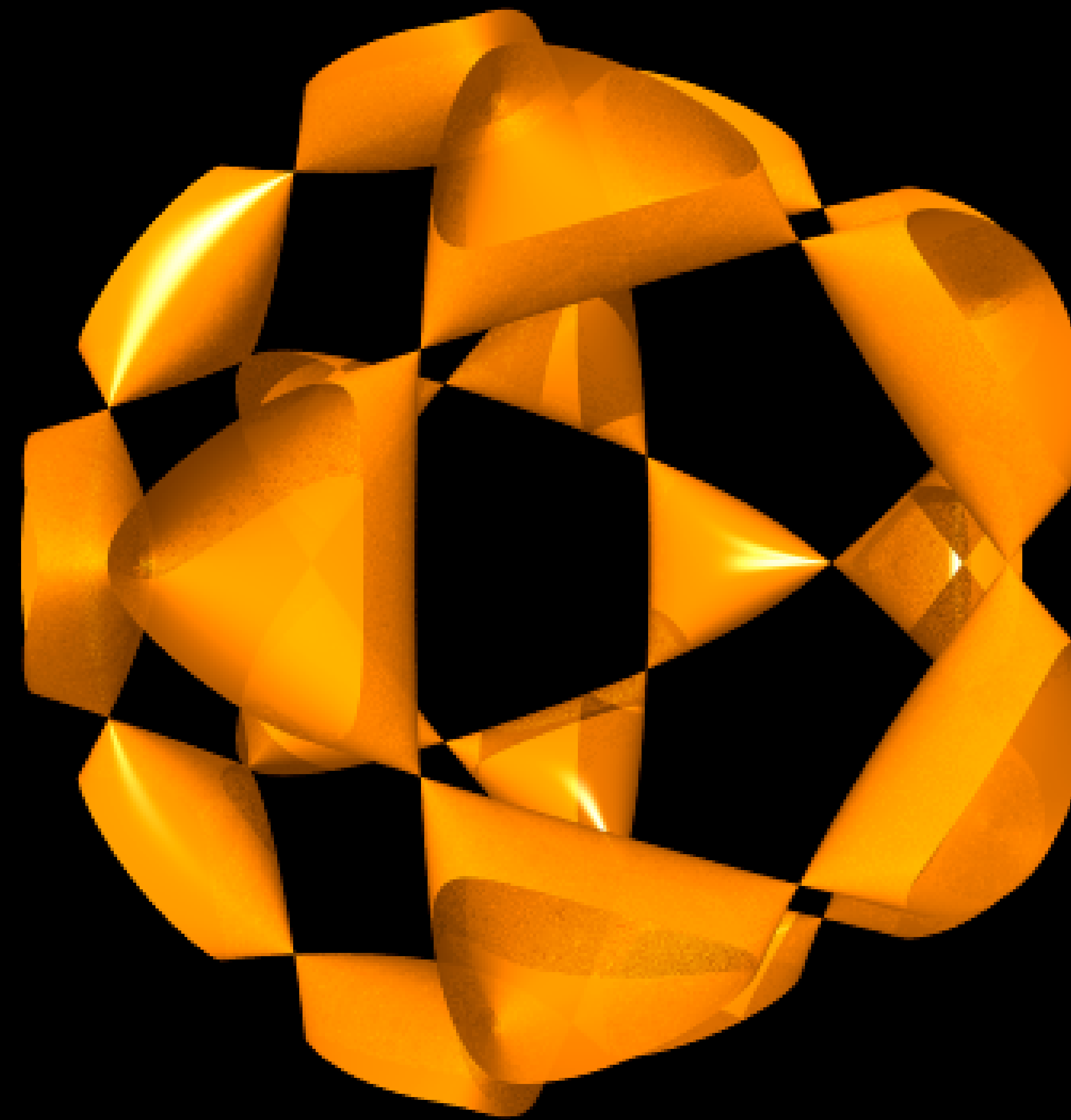
The Weather Channel
weather.com
15 May 2006 14:28 GMT / 15 May 2006 10:28 AM EDT



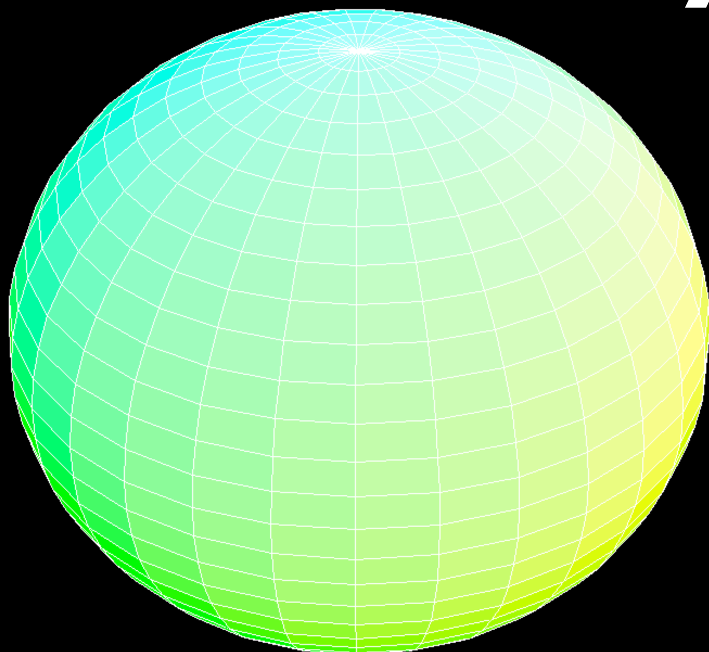
Surfaces

2 ways to represent surfaces

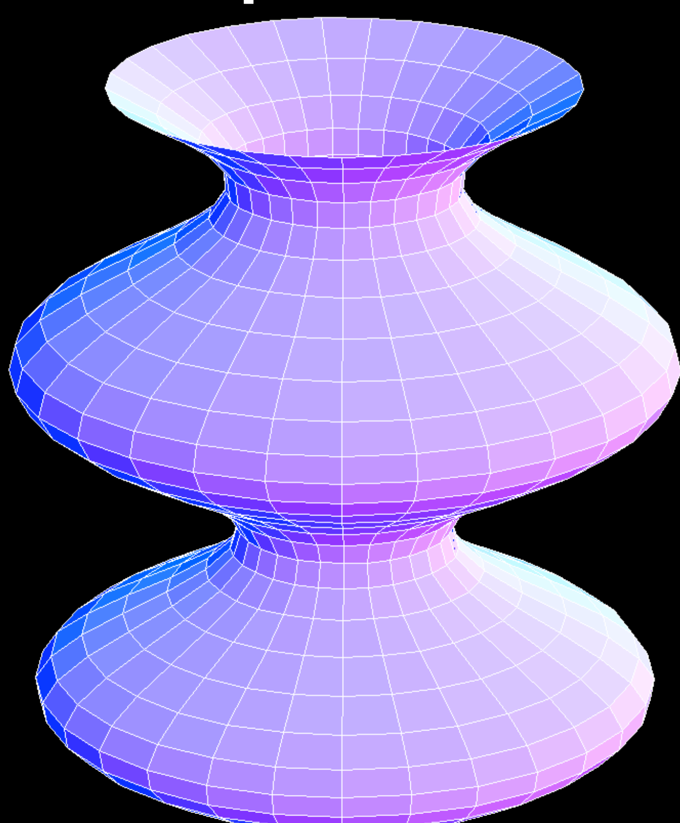


- Implicit surface g(x,y,z)=c
- Parametric surface r(u,v)

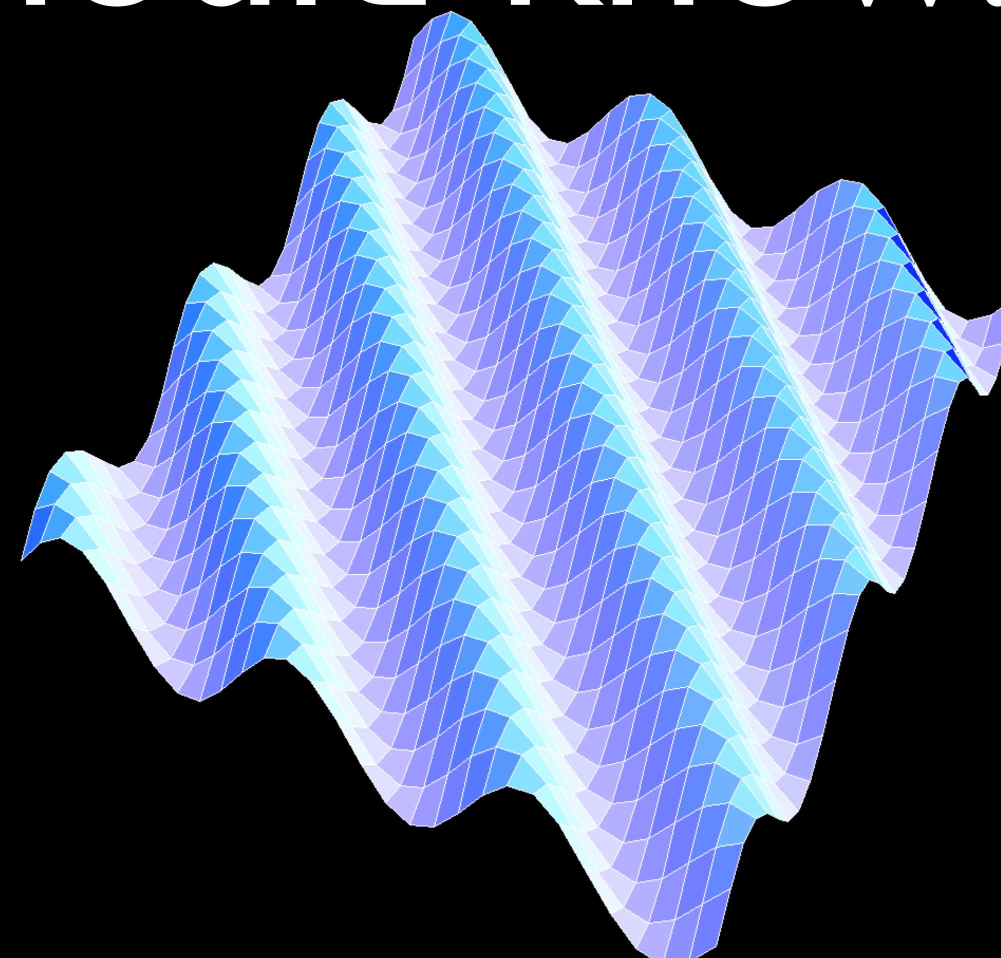
Surfaces you should know:



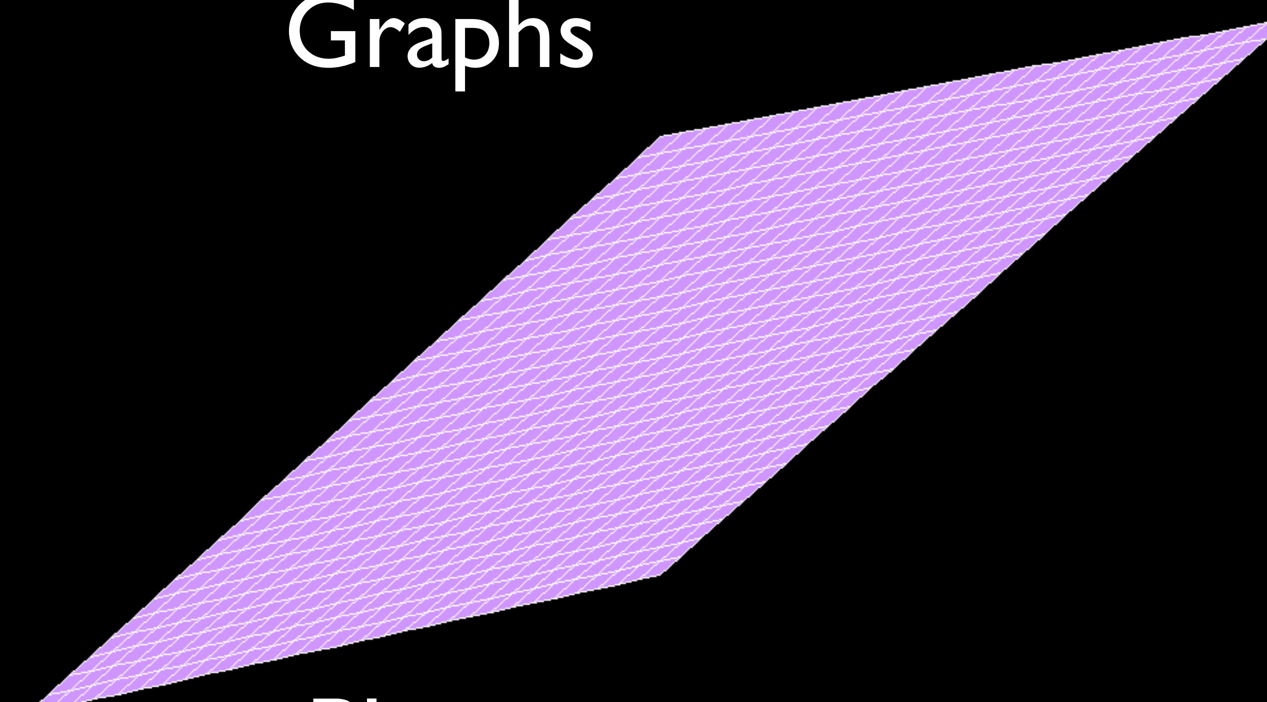
Spheres



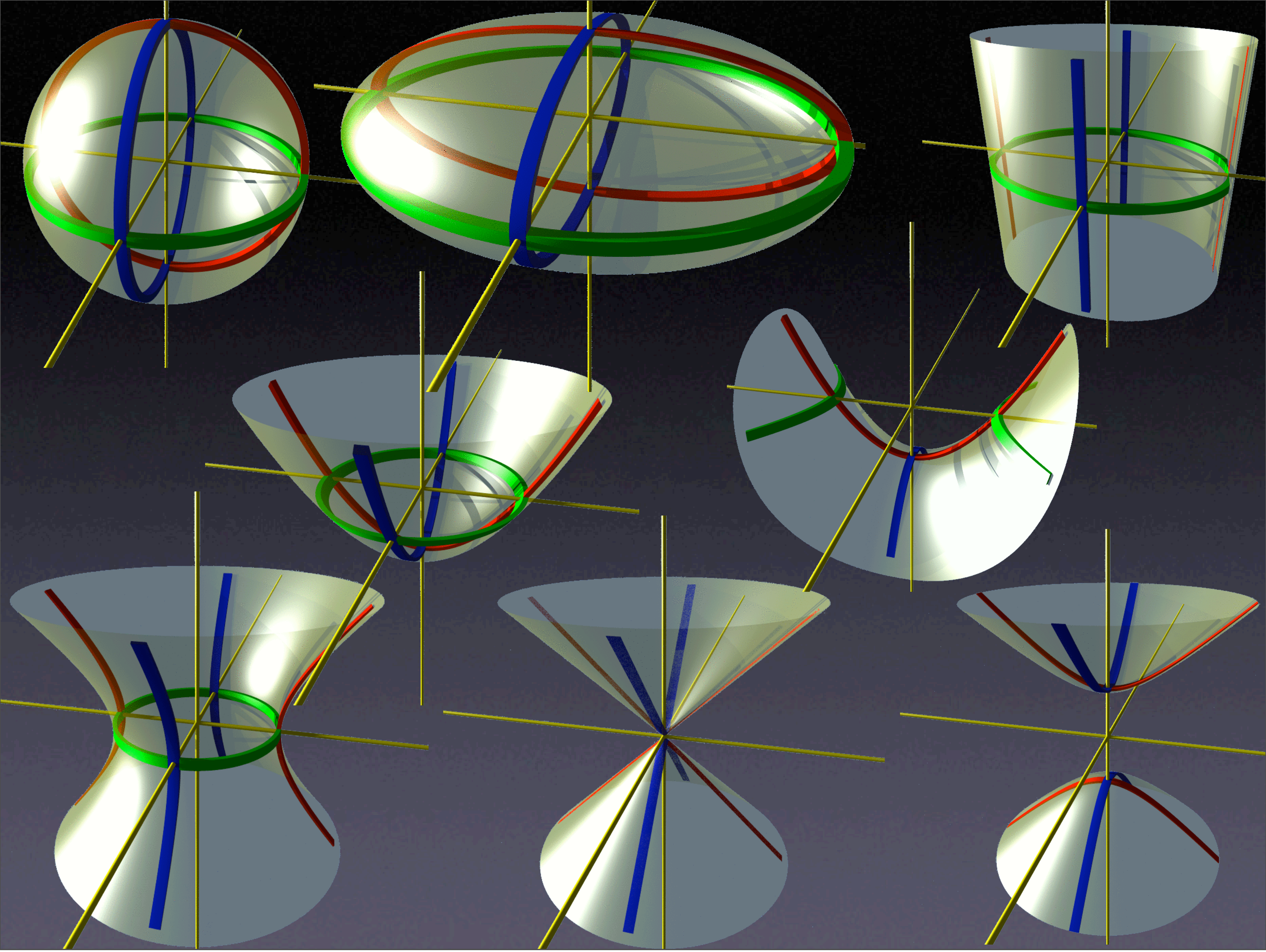
Surfaces of revolution



Graphs

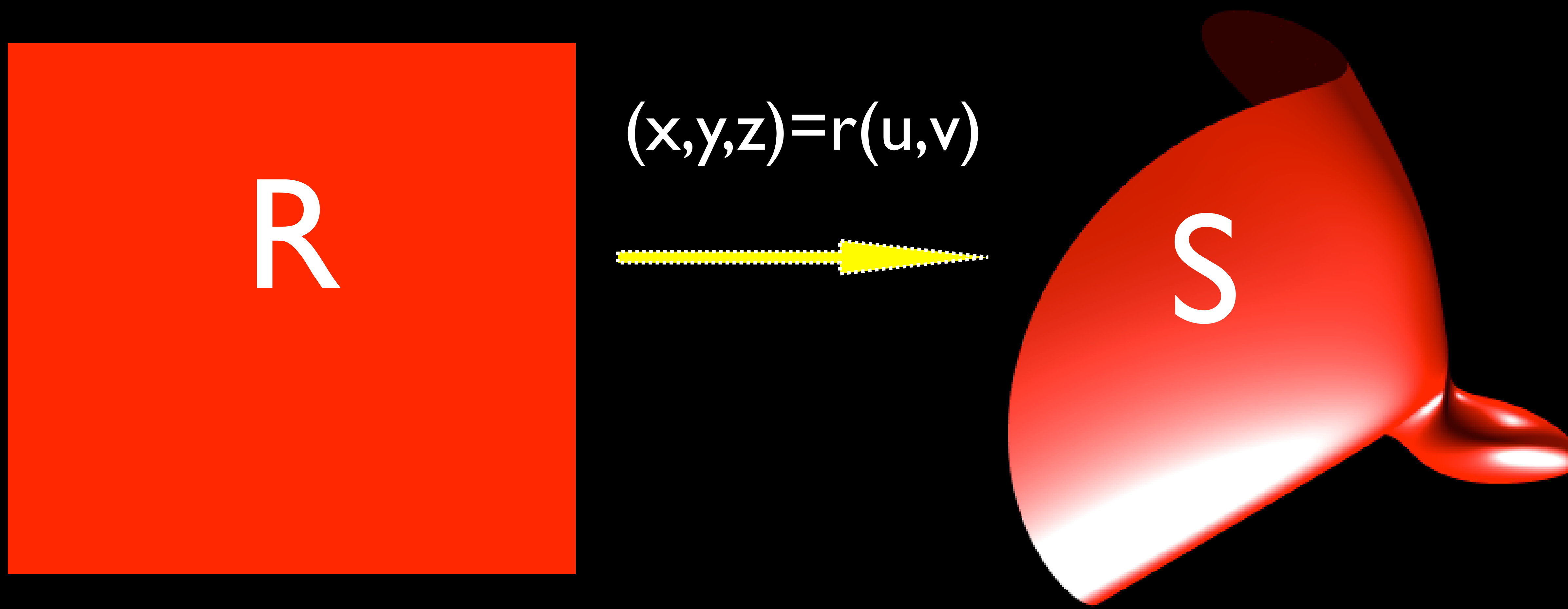


Planes



Surface Area

$$r(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$r(R) = S$$
$$\text{Area}(S) = \int\int_R |r_u \times r_v| \, du dv$$



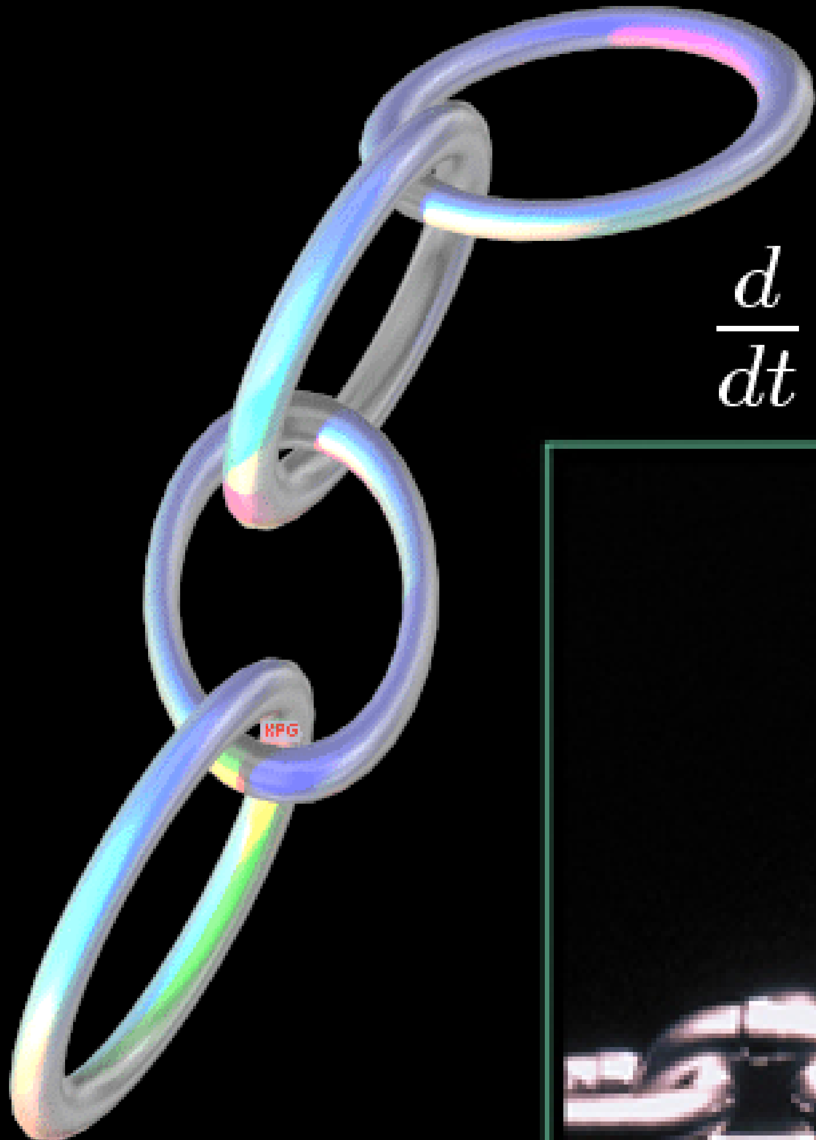
Thats all I have to say about that.

Derivatives

$$\partial_x f = f_x = \frac{\partial f}{\partial x}$$

$$\nabla f(x, y, z) = (f_x, f_y, f_z) = \mathrm{grad}(f)$$

$$D_v f(x, y, z) = \nabla f(x, y, z) \cdot v$$



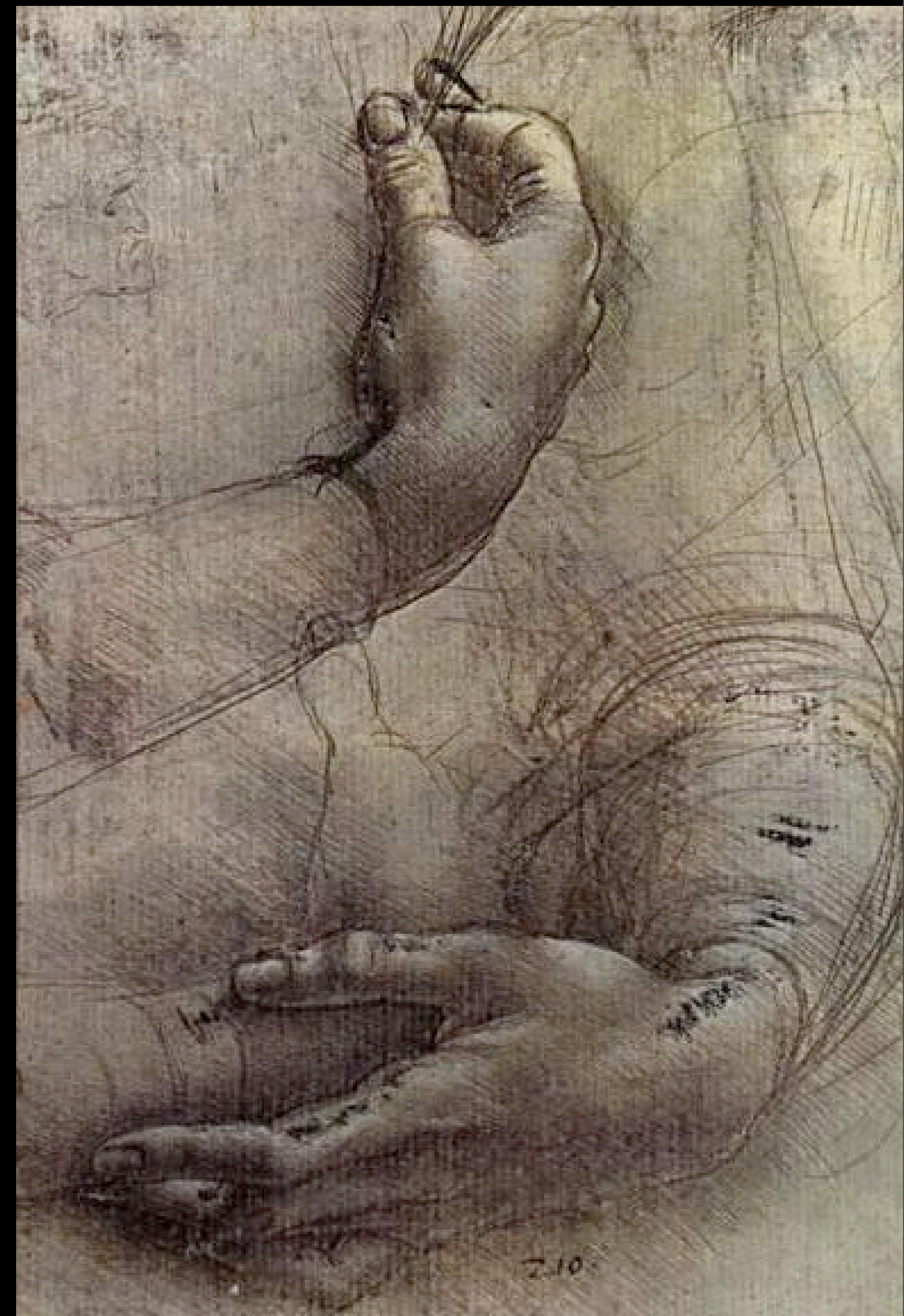
Chain Rule

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

Implicit Differentiation

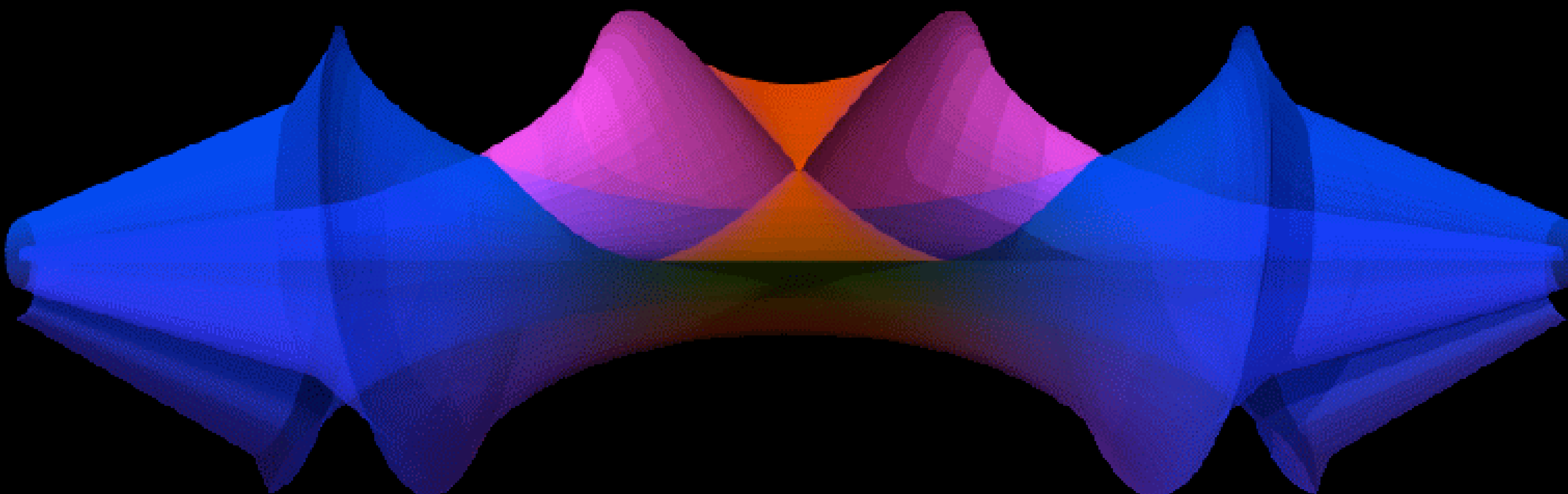
g(x,y,z) = 0 defines z = f(x,y)

$$z_x = -g_x(x, y, z)/g_z(x, y, z)$$

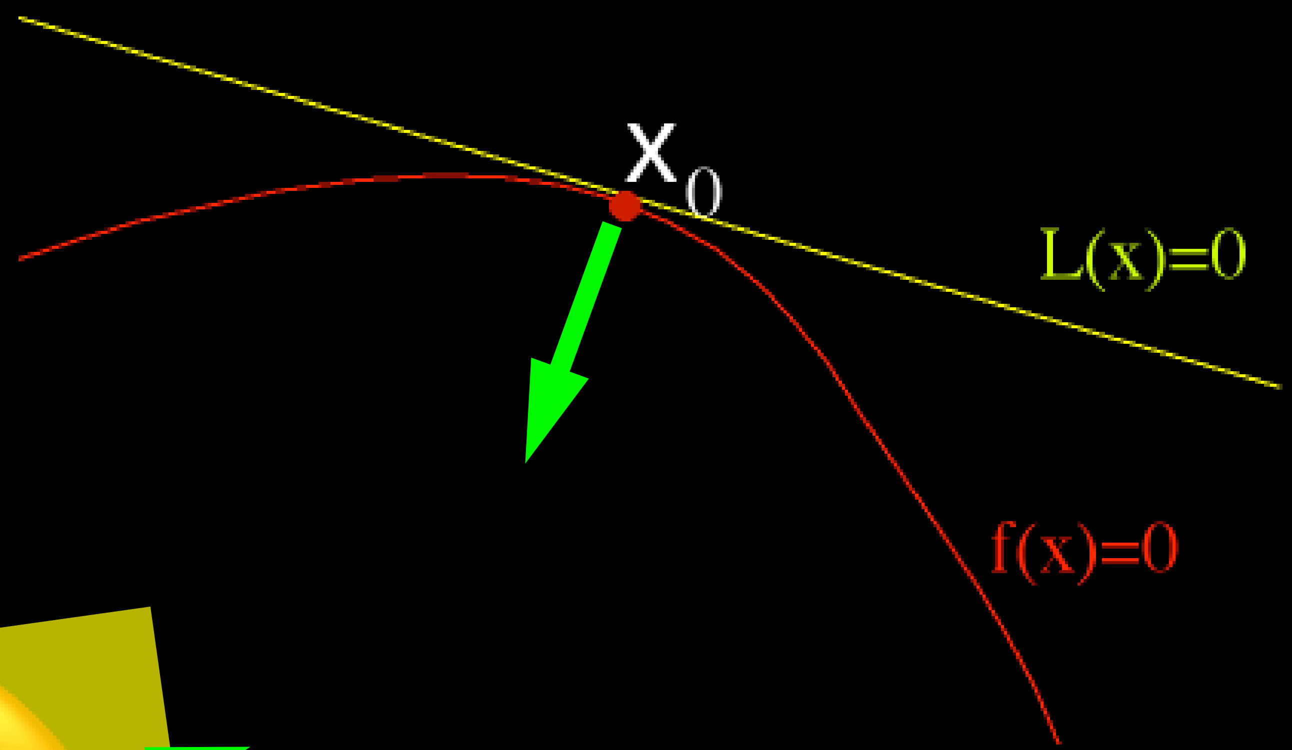


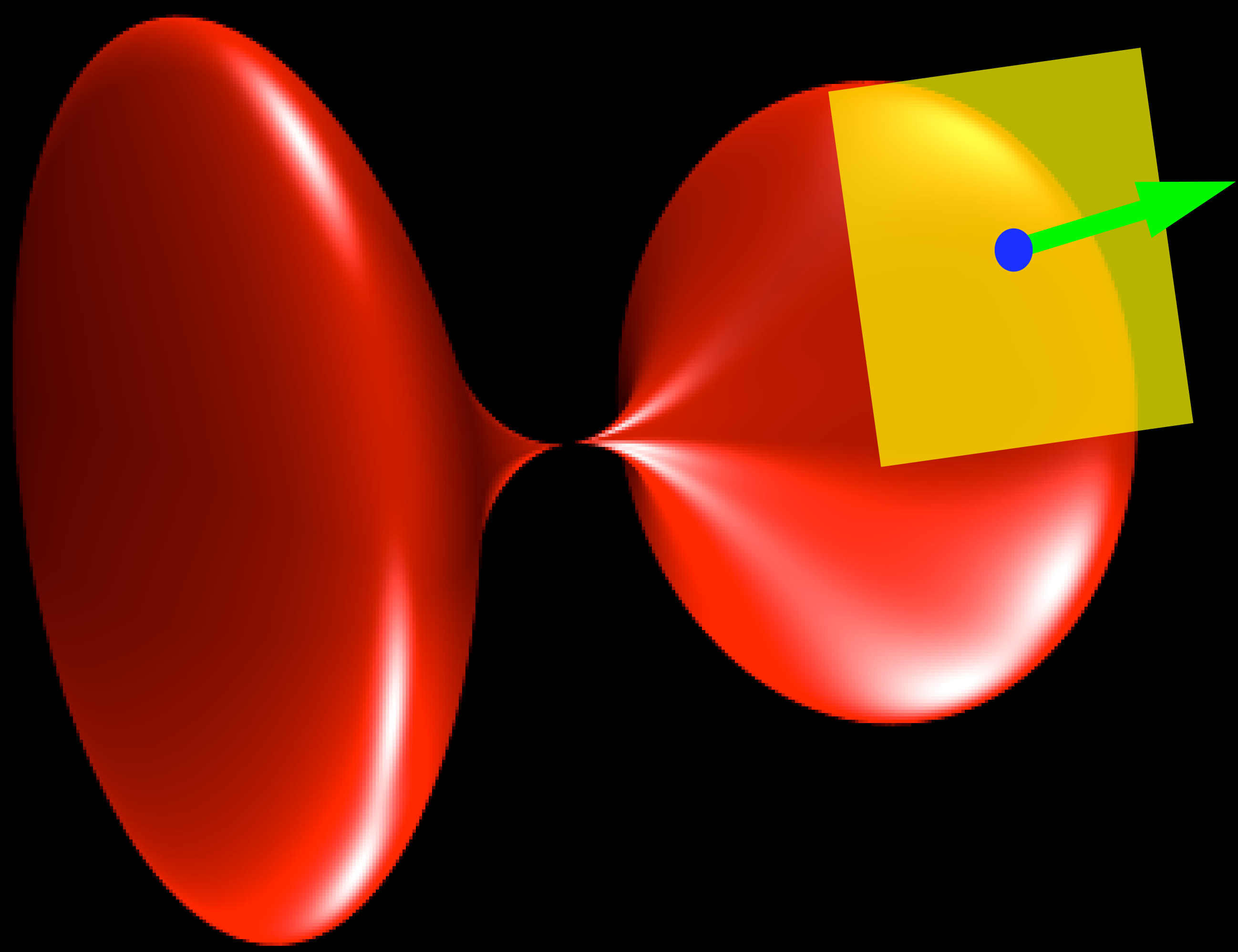
Example:

$$x^5 + y^5 - z - z^7 - 1 = 0$$
$$z = g(x, y)$$
$$g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z) = -5x^4/(1 + 7z^6)$$
$$g_x(1, 0) = -5$$



Gradients and Tangents





Crucial: the normal vector (a,b,c) is the gradient. The equation of the plane is

ax+by+cz=d

where d is found at the end.

A Trinity:

Linear Approximation

Estimation

Tangent plane

Richard Feynmans trick

Movie: "Infinity" (1996)

Estimation

The cube root of 1729.02 is close
to the cube root of 1728, which is 12=x0.

We have

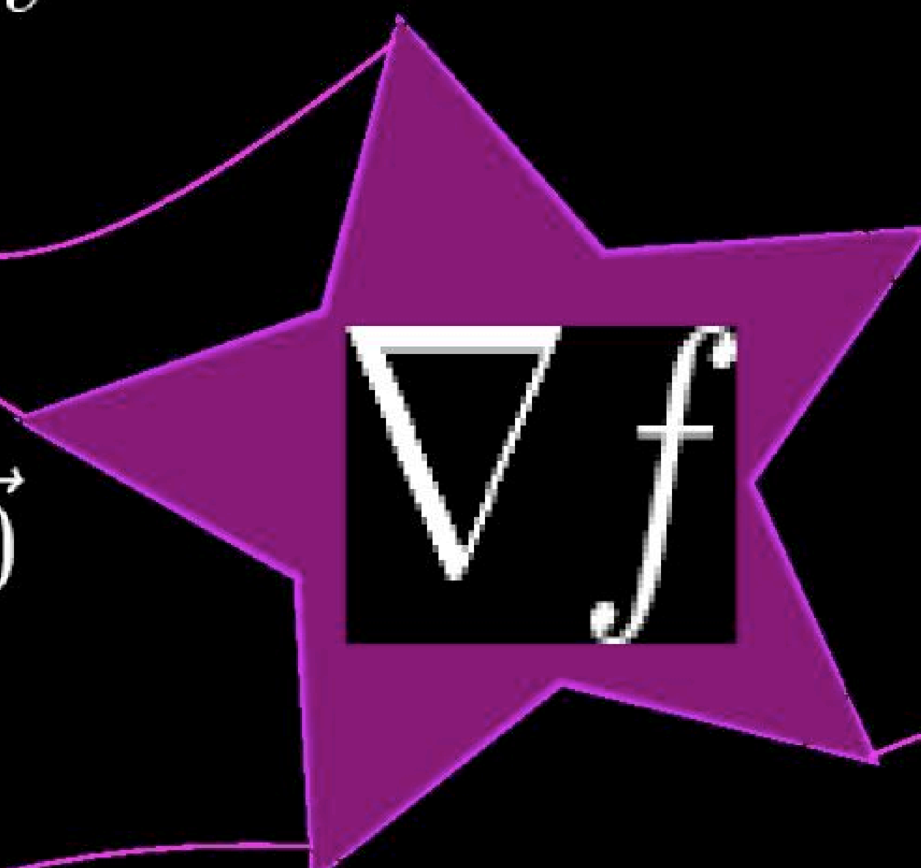
f(x)=f(1728)+f'(1728) 1.02 = 12+1.02/(144 *3)
=12.00236

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$\nabla f = \vec{0}$$



$$\nabla f(\vec{x}_0) = (a, b, c)$$

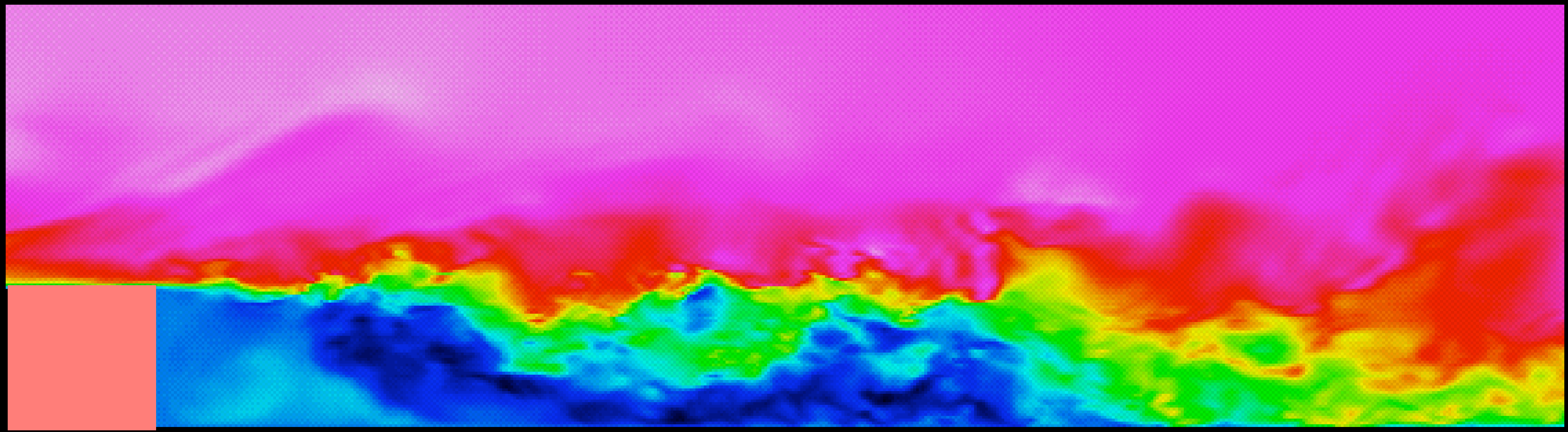
$ax + by + cz = d$ tangent plane

$f(x, y) \sim L(x, y)$ near (x_0, y_0)

PDE's

Equations in which partial derivatives appear: PDE's

$$\frac{d}{dt}u + u \cdot \nabla u = \nu \Delta u - \nabla p + f$$
$$\mathrm{div} u = 0$$



PDE's

PDE examples appearing in this course are Clairot's theorem:

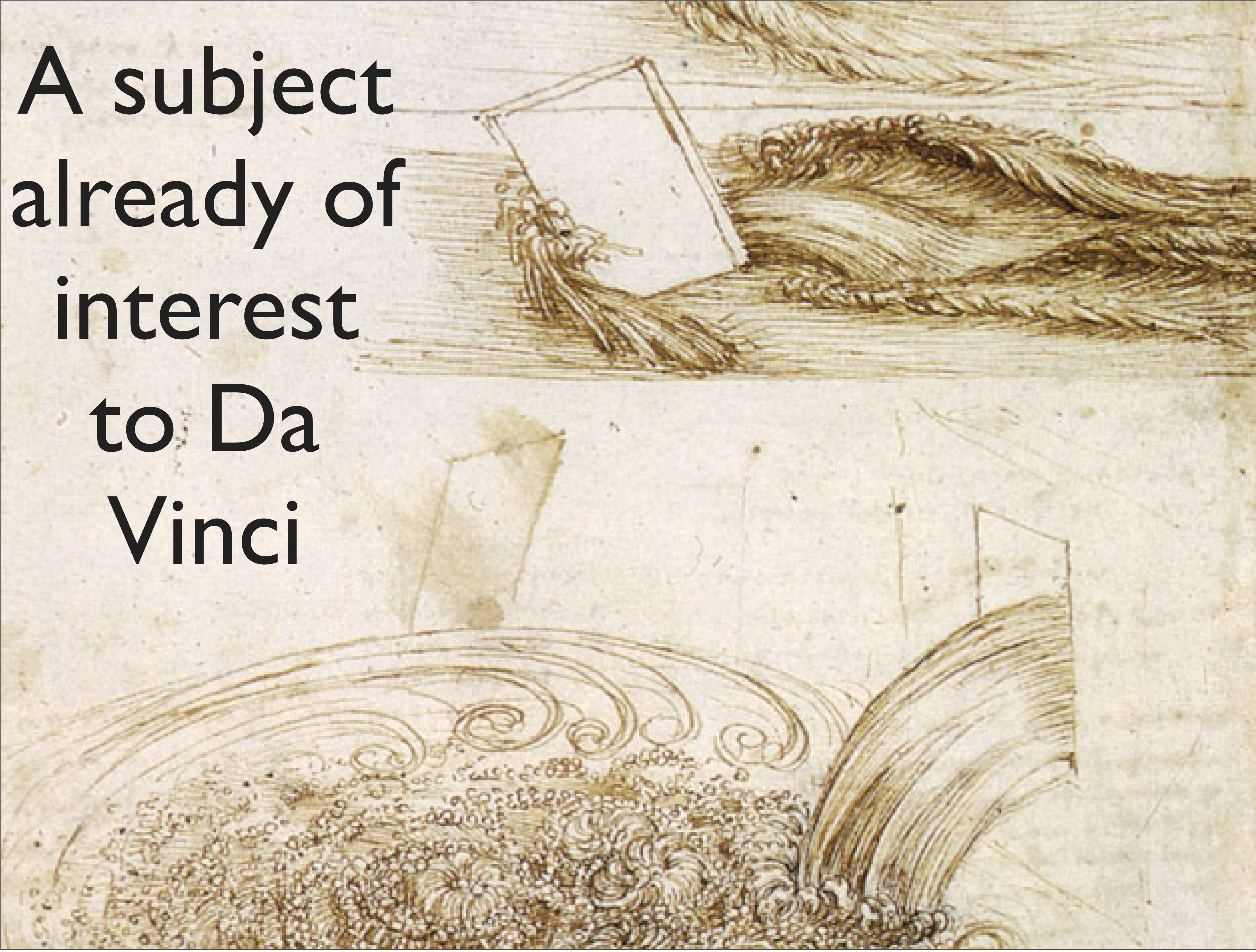
$$f_{xy} - f_{yx} = 0.$$

or identities like curl (grad(f))=0
or div(curl(F))=0 which are
based on Clairot.

PDE's

You should be able to verify that certain functions satisfy a PDE like

$u_{xx} + u_{yy} = c^2 u_{tt}$	wave
$\mu(u_{xx} + u_{yy}) = u_t$	heat
$u_t = c u_x$	transport

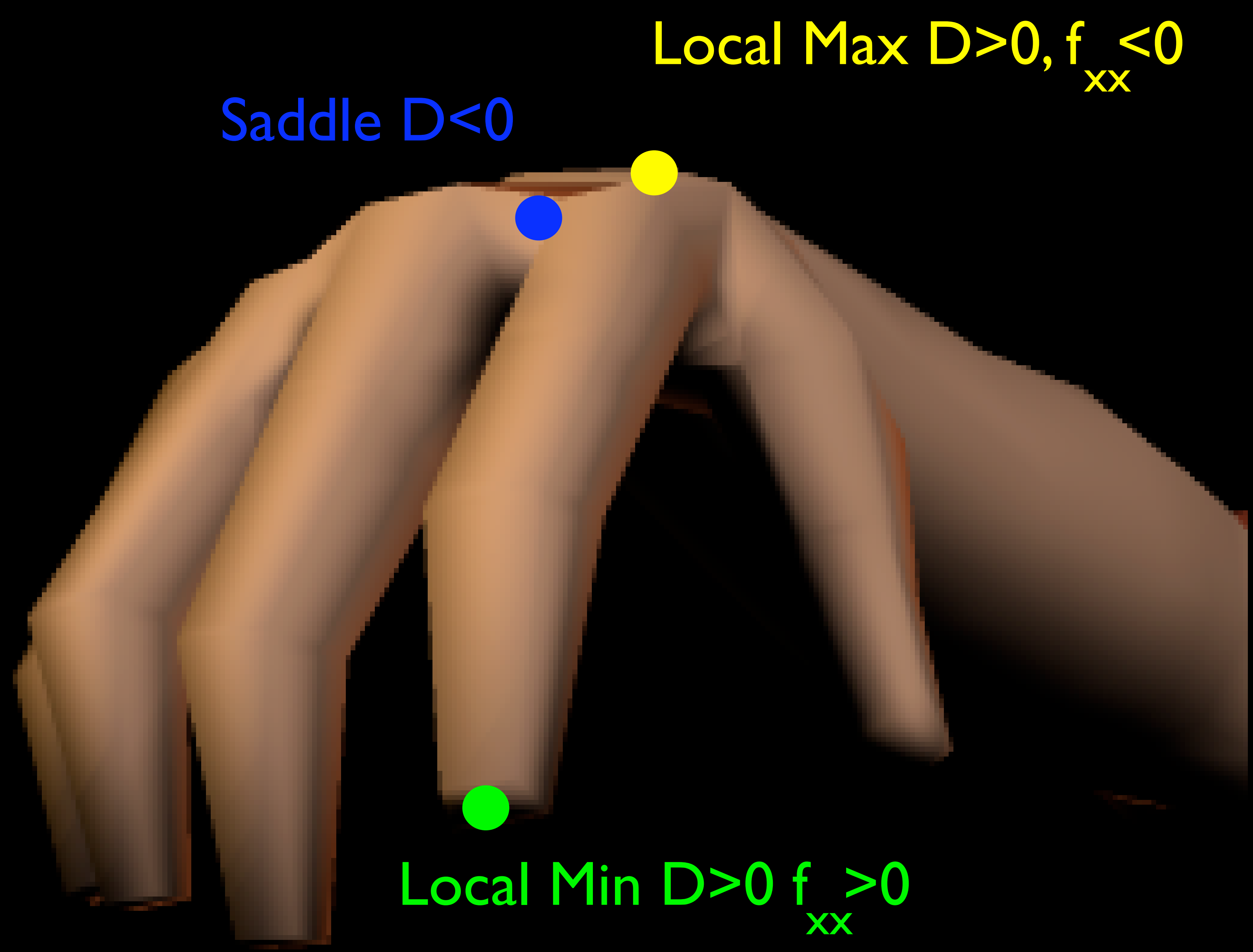


A subject already of interest to Da Vinci

Thats all I have to say about that.

Extrema

Second Derivative Test



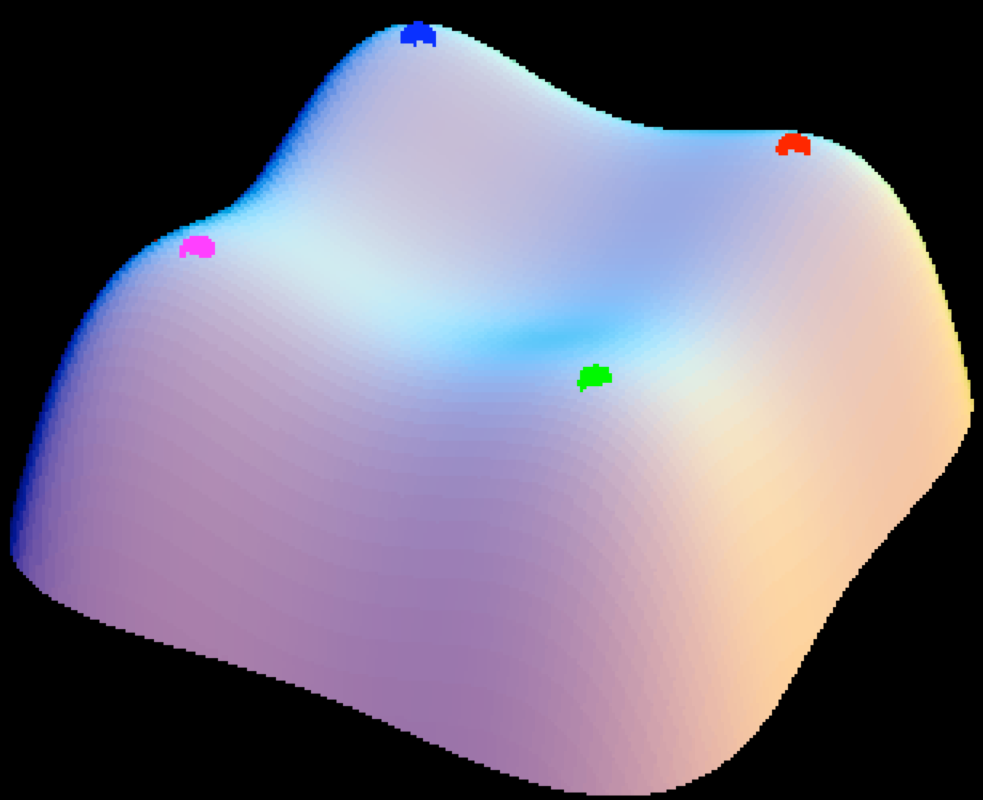
Problem 6: Find all critical points of

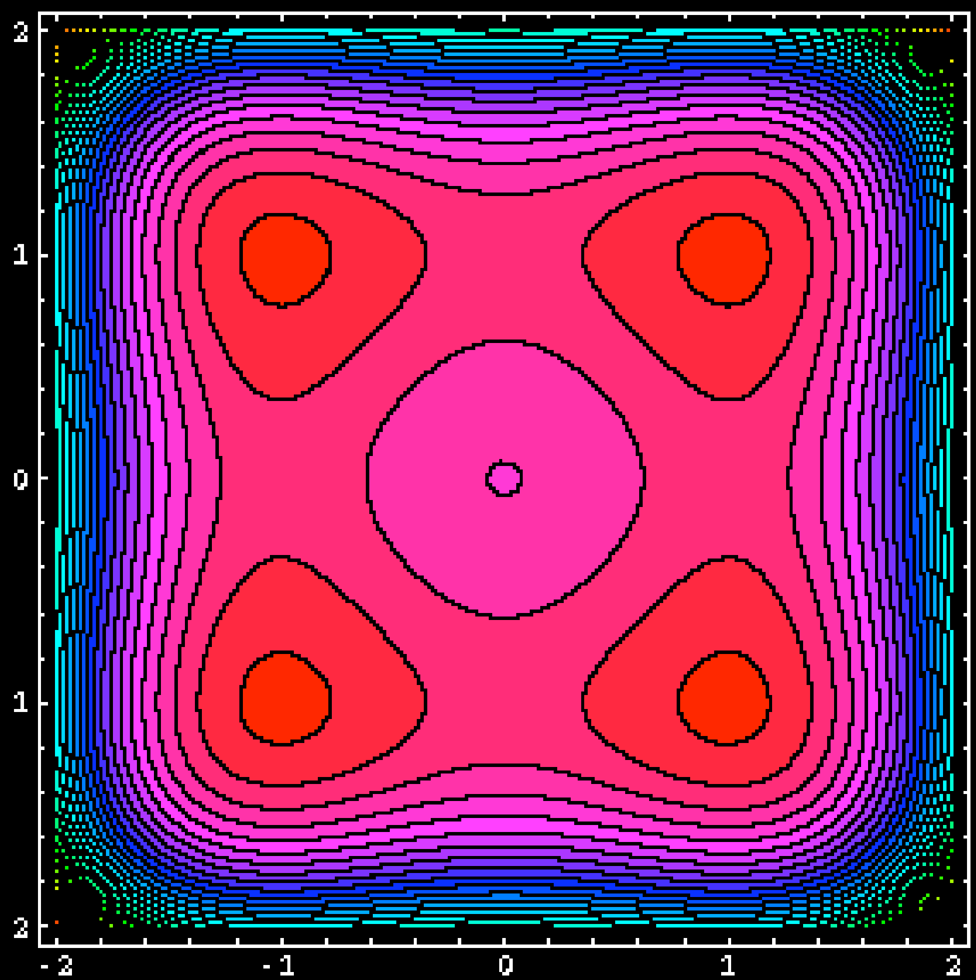
$$f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$$

and classify them.

Problem 5: extremize

$$f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$$

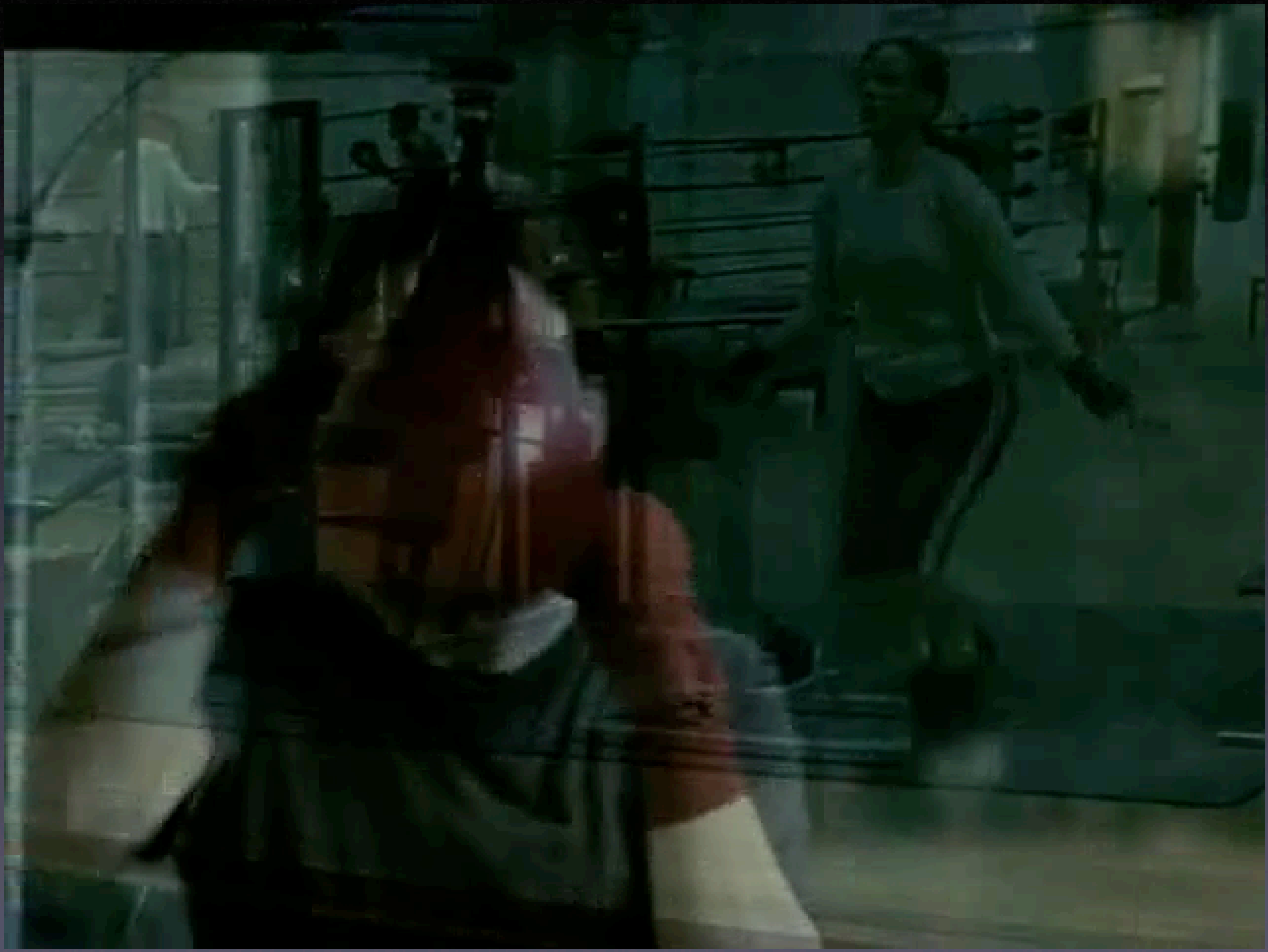




Quiz coming up

Win a DVD.

DVD preview



Are you ready for a fight with this problem?

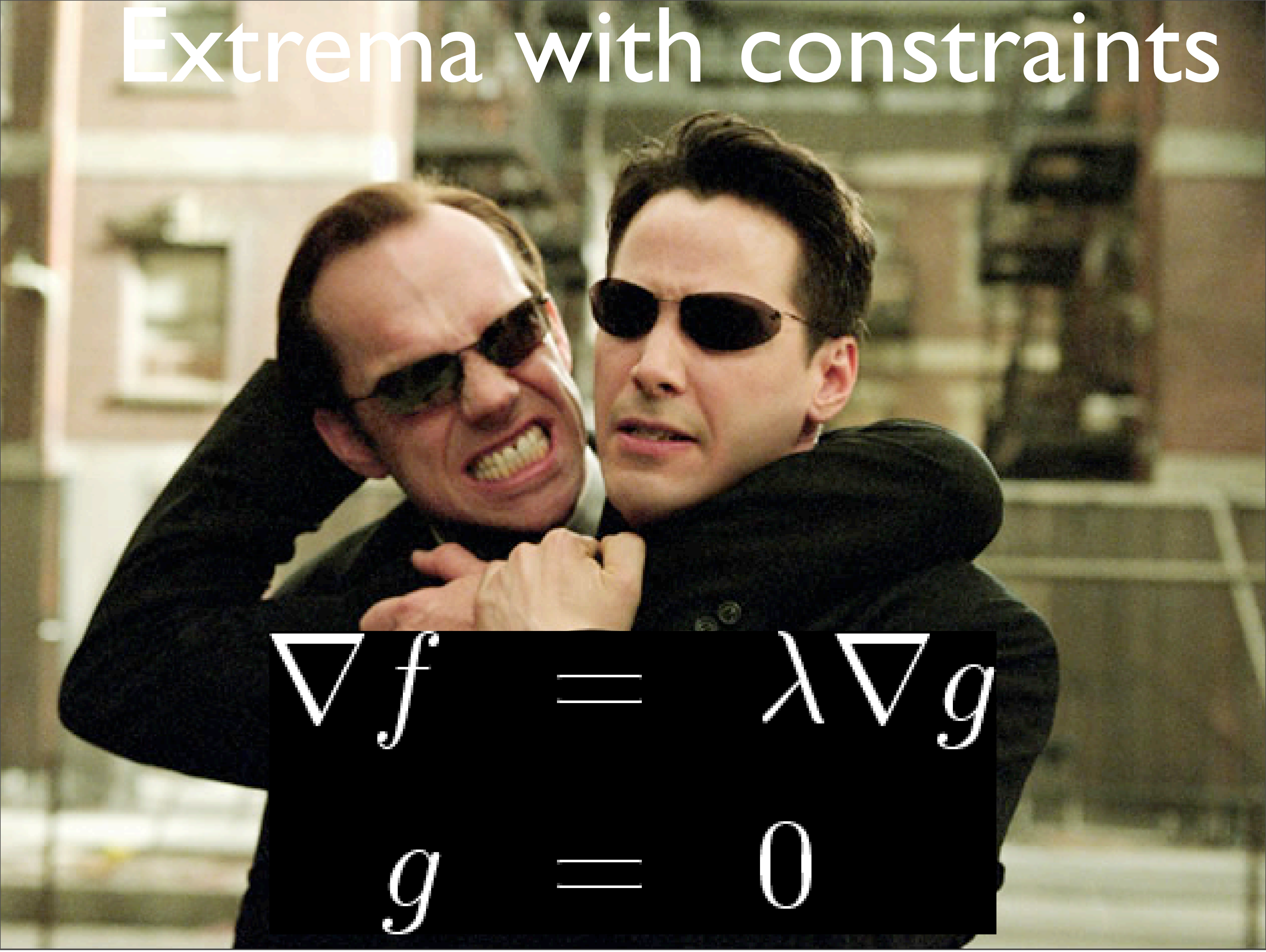
You know that the Laplacian of f(x,y) is negative everywhere. What can you say about the critical points of f?

Answer:

Every critical point is a saddle point if $f_{yy} > 0$ because $D < 0$ in that case.

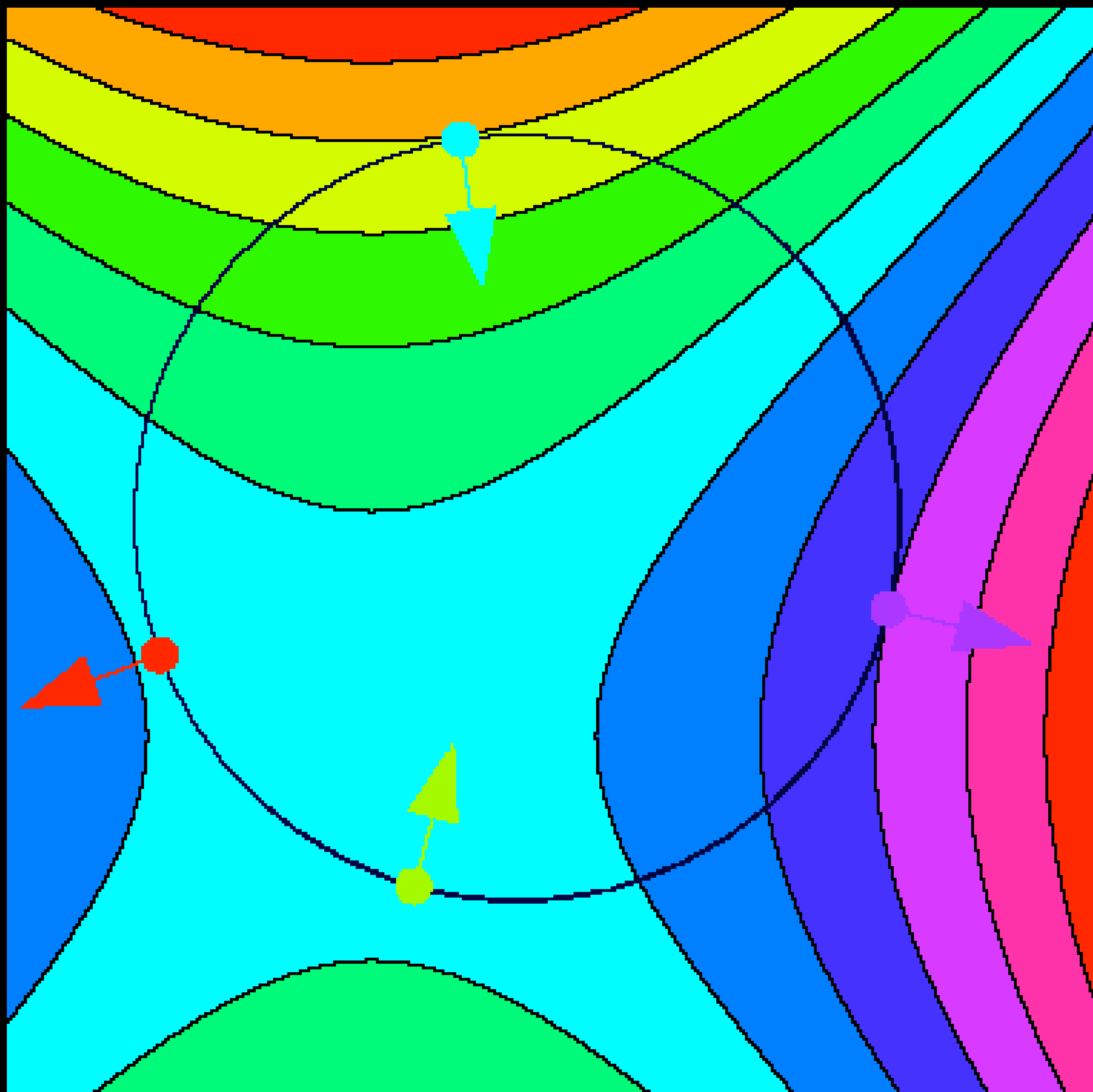
If $f_{yy} < 0$, then the critical point can be either a saddle point or a local min.

Extrema with constraints



$$\nabla f = \lambda \nabla g$$
$$g = 0$$

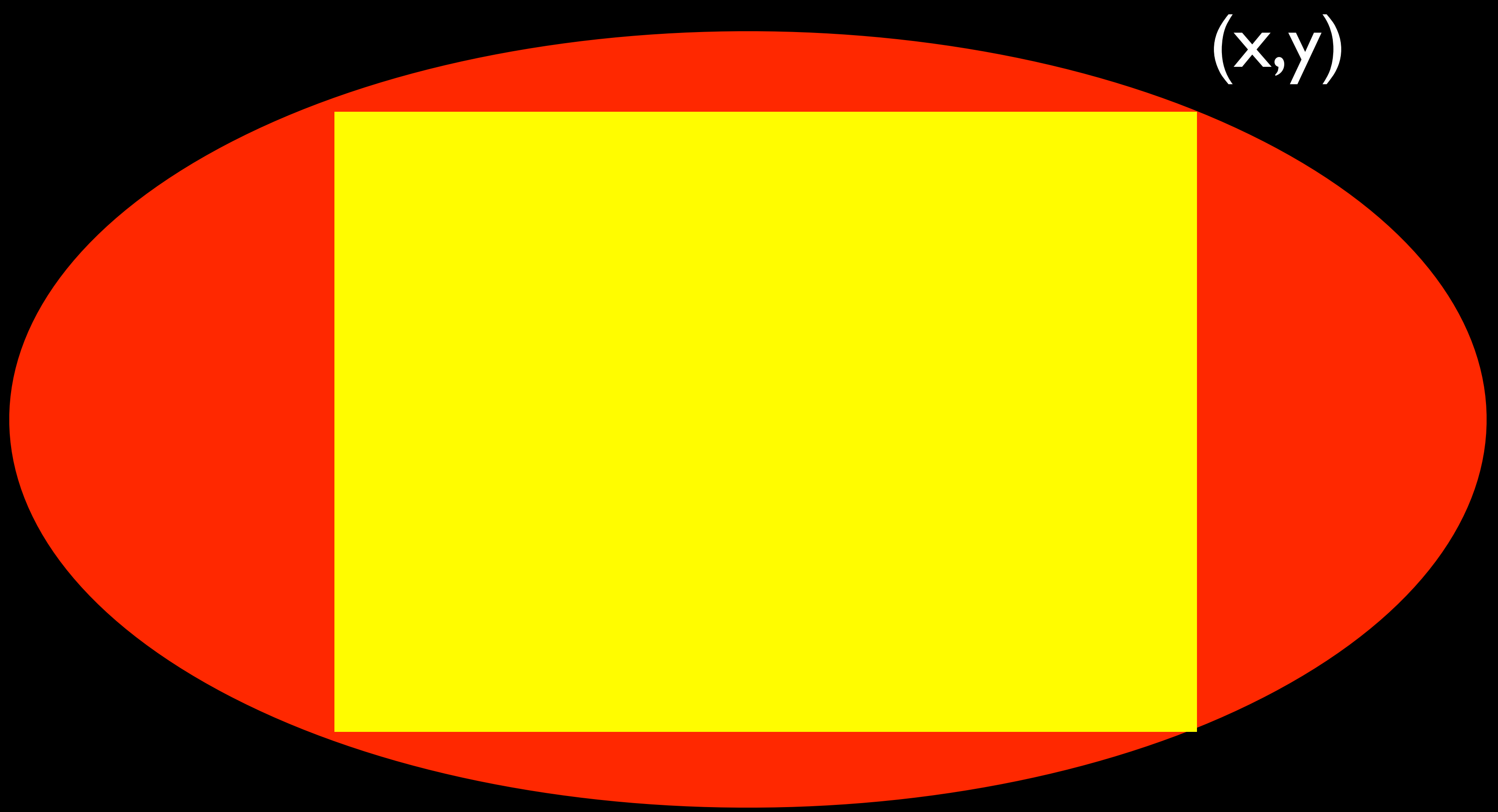
Lagrange



$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= 0 \end{aligned}$$

Problem

Find the largest rectangle inscribed in the ellipse E

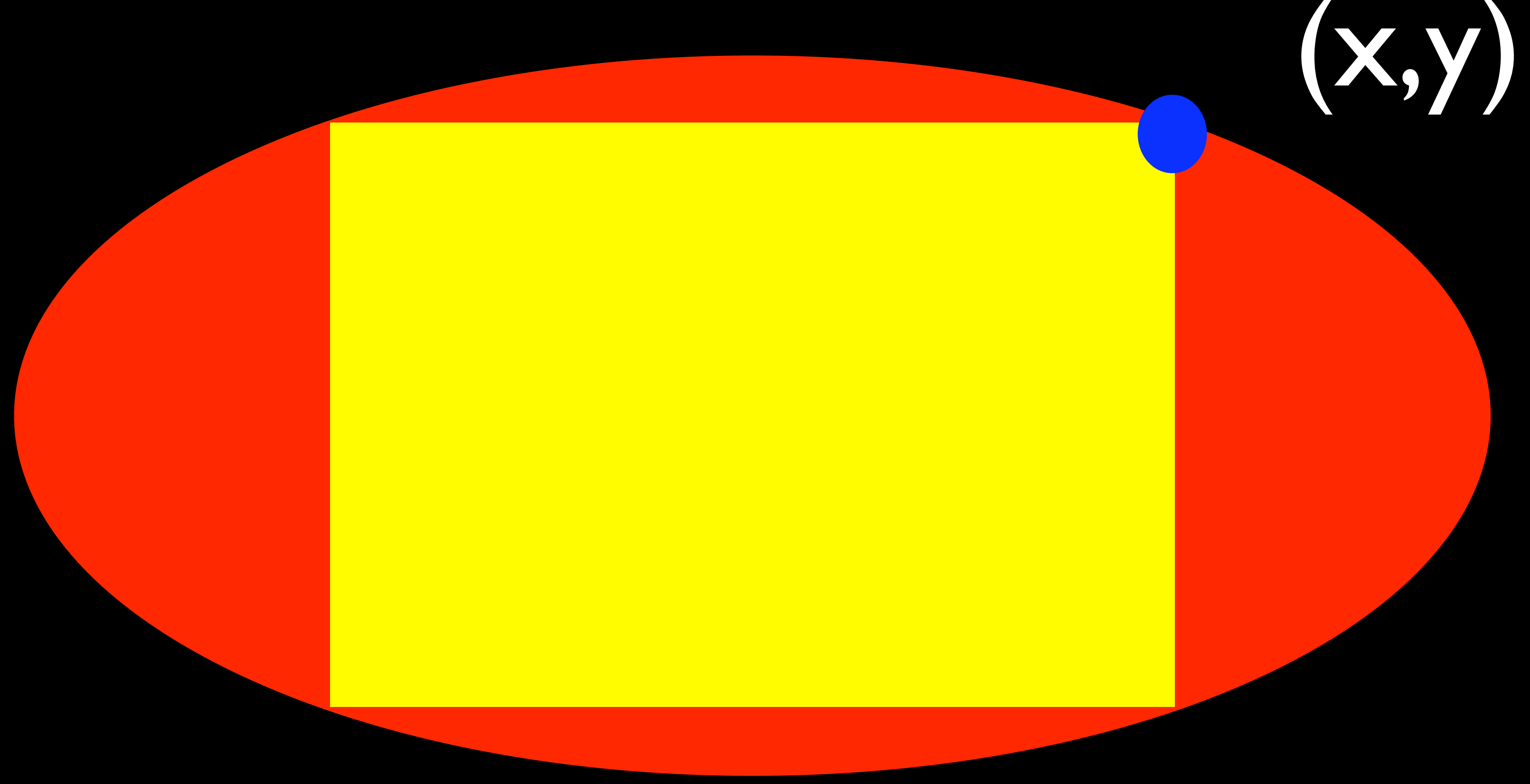


Ellipse E: $$\frac{x^2}{4} + y^2 = 1$$

Lagrange Problem

$$f(x,y) = 4xy$$

$$g(x,y) = x^2/4 + y^2 - 1 = 0$$



Ellipse E: $\frac{x^2}{4} + y^2 = 1$

Solution

$$f(x,y) = 4xy$$

$$g(x,y) = x^2/4 + y^2 - 1 = 0$$

Lagrange equations

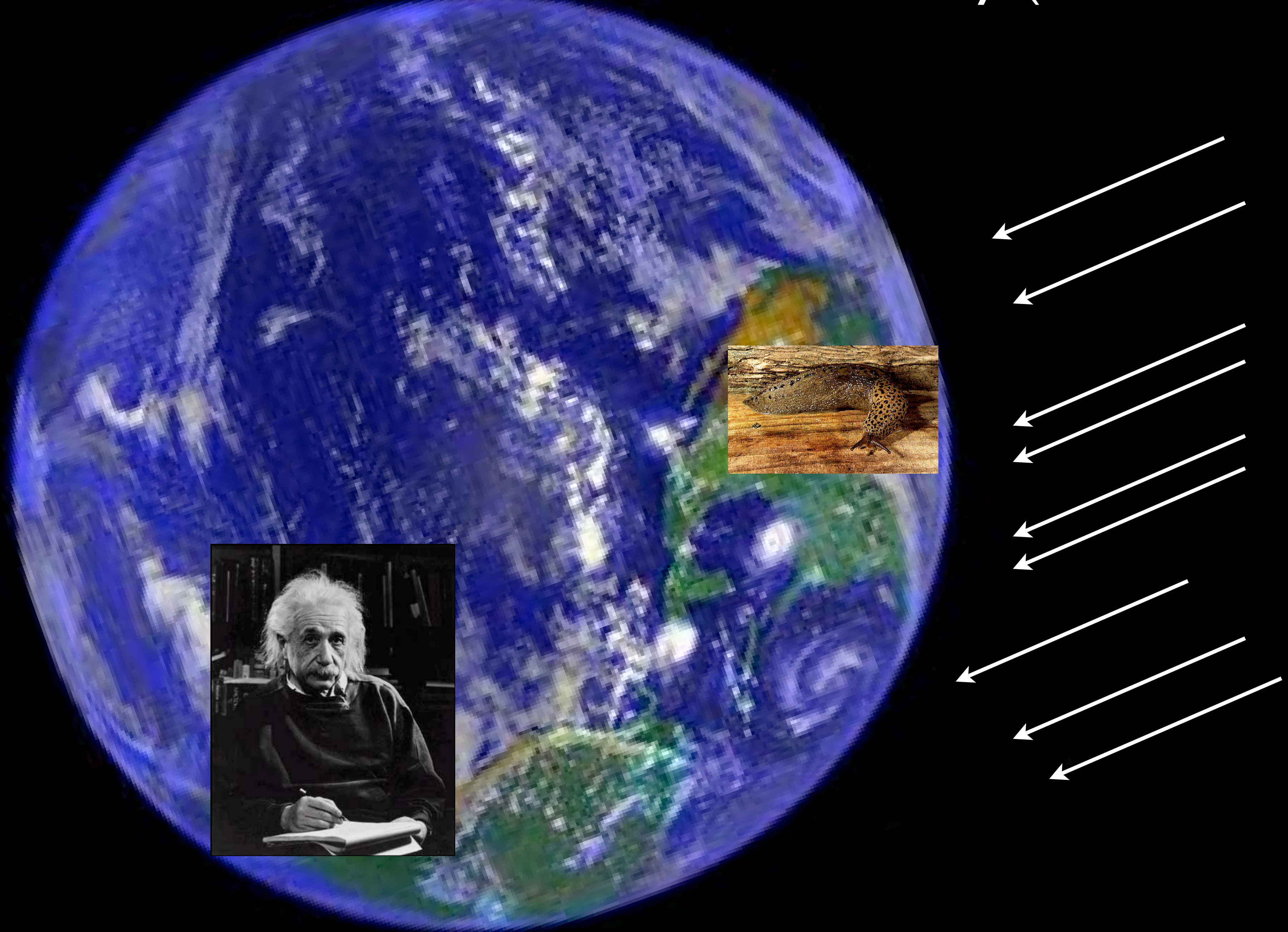
$$4y = \lambda x/2$$
$$4x = \lambda 2y$$
$$g(x,y) = 0$$

$$x = 2y, \frac{x^2}{4} + y^2 = 1$$

$$\Rightarrow y = 1/\sqrt{2}, x = \sqrt{2}$$

A final problem:

The earths surface is the unit sphere. An alian species shines a mental enhancement ray in our direction with intensity $f(x,y,z) = xy + z$. Where is the maximal intensity on earth (we become Einsteins) where is the minimal intensity (we become slugs)



Thats all I have to say about that.

An exam is like a box of chocolates. You never know what you are gonna get.

Part II

To-
morrow
at the
same
time.