

Name:

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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.
- Please write neatly.
- Do not use notes, books, calculators, computers, or other electronic aids.
- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.
- You have 180 minutes time to complete your work.
- The Biochem section can ignore problems with vector fields and line integrals.

MWF 10 Samik Basu
MWF 10 Joachim Krieger
MWF 11 Matt Leingang
MWF 11 Veronique Godin
TTH 10 Oliver Knill
TTH 115 Thomas Lam

1		20
2		10
3		10
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12A		10
13A		10
14A		10

12B		10
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14B		10

Problem 1) True/False questions (20 points)

- 1)

T	F
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 For any two nonzero vectors \vec{v}, \vec{w} the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v}$ is parallel to \vec{w} .
- 2)

T	F
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 The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.
- 3)

T	F
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 If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t , then the curve is part of a line.
- 4)

T	F
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 The curve $\vec{r}(t) = (1 - t)A + tB, t \in [0, 1]$ connects the point A with the point B .
- 5)

T	F
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 For every c , the function $u(x, t) = (2 \cos(ct) + 3 \sin(ct)) \sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.
- 6)

T	F
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 The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1 + \cos^2(t)} dt$.
- 7)

T	F
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 Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.
- 8)

T	F
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 The function $f(x, y, z) = x^2 - y^2 - z^2$ decreases in the direction $(2, -2, -2)/\sqrt{8}$ at the point $(1, 1, 1)$.
- 9)

T	F
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 Assume \vec{F} is a vector field satisfying $|\vec{F}(x, y, z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0, 1]$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is less or equal than the arc length of C .
- 10)

T	F
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 Let \vec{F} be a vector field which coincides with the unit normal vector \vec{N} for each point on a curve C . Then $\int_C \vec{F} \cdot d\vec{r} = 0$.
- 11)

T	F
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 If for two vector fields \vec{F} and \vec{G} one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a, b, c)$, where a, b, c are constants.
- 12)

T	F
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 If a nonempty quadric surface $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.
- 13)

T	F
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 If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) , then $\text{curl}(\vec{F}) = (0, 0, 0)$ for all (x, y, z) .
- 14)

T	F
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 If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

TF PROBLEMS FOR REGULAR AND PHYSICS SECTIONS:

- 15)

T	F
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 The divergence of the gradient of any $f(x, y, z)$ is always zero.
- 16)

T	F
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 For every vector field \vec{F} the identity $\text{grad}(\text{div}(\vec{F})) = \vec{0}$ holds.
- 17)

T	F
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 For every function f , one has $\text{div}(\text{curl}(\text{grad}(f))) = 0$.
- 18)

T	F
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 If \vec{F} is a vector field in space then the flux of \vec{F} through any closed surface S is 0.
- 19)

T	F
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 The flux of the vector field $\vec{F}(x, y, z) = (y + z, y, -z)$ through the boundary of a solid region E is equal to the volume of E .

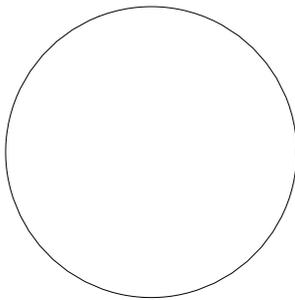
- 20) T F For every function $f(x, y, z)$, there exists a vector field \vec{F} such that $\text{div}(\vec{F}) = f$.

TF PROBLEMS FOR BIOCHEM SECTIONS:

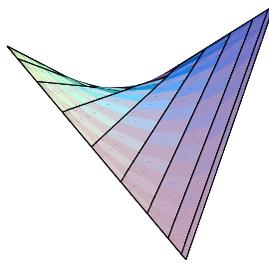
- 21) T F The following reasoning is correct: tossing 3 unbiased coins, the possible numbers of heads appearing are 0, 1, 2, and 3. Therefore each of these events has probability $1/4$.
- 22) T F Two events A, B for which $P(B) > 0$ are independent if and only if $P(A|B) = P(A)$.
- 23) T F For two independent random variables X, Y one has the following identities for the variance $D(X) - D(Y) = D(X - Y)$.
- 24) T F Let A, B be arbitrary events. If $P(A|B) = P(B|A)$ then $P(A) = P(B)$.
- 25) T F The probability that from 6 random coins tossed, 6 show tail is smaller than the probability that 5 show tail. Each coin has probability $1/2$ to show tail.
- 26) T F If you throw 2 dice and you know the first one shows the number 1, then the chance that the second one shows 1 is less than $1/6$.

Problem 2) (10 points)

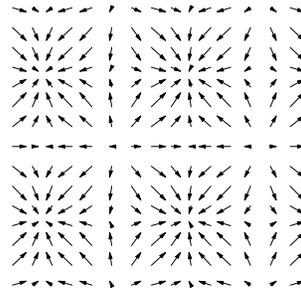
Match the equations with the objects. No justifications are needed.



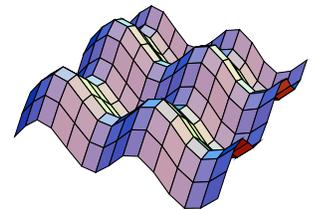
I



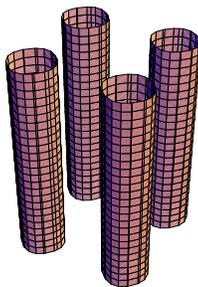
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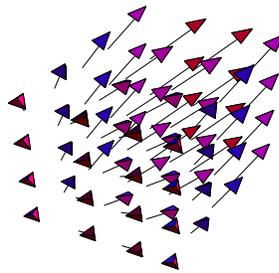
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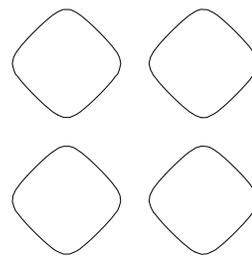
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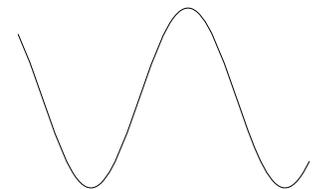
V



VI



VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x, y, z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = (\cos(t), \sin(t))$
	$\vec{r}(u, v) = (\cos(u), \sin(v), \cos(u) \sin(v))$
	$\vec{F}(x, y, z) = (\cos(x), \sin(x), 1)$
	$z = f(x, y) = \cos(x) + \sin(y)$
	$g(x, y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x, y) = (\cos(x), \sin(x))$

Problem 3) (10 points)

Mark with a cross in the column below "conservative" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x, y, z) = (0, 0, 0)$ for all points (x, y, z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z)). No justifications are needed.

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$		
$\vec{F}(x, y, z) = (x, y, z)$		
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$		

Problem 4) (10 points)

Let E be a parallelogram in three dimensional space defined by two vectors \vec{u} and \vec{v} .

- a) (3 points) Express the diagonals of the parallelogram as vectors in terms of \vec{u} and \vec{v} .
- b) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?
- c) (4 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} dx dy.$$

Problem 7) (10 points)

In this problem we evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} dx dy$, where D is the triangular region bounded by the x and y axis and the line $x + y = 1$.

- a) (3 points) Find the region R in the uv -plane which is transformed into D by the change of variables $u = x - y, v = x + y$. (It is enough to draw a carefully labeled picture of R .)
- b) (3 points) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $(x, y) = (\frac{u+v}{2}, \frac{v-u}{2})$.
- c) (4 points) Evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} dx dy$ using the above defined change of variables.

Hint. The general topic of change of variables does not appear this semester. You can solve the problem nevertheless, when given the formula $\frac{\partial(x,y)}{\partial(u,v)} = x_y y_v - x_v y_u$ for the integration factor (analogous to r when changing to polar coordinates, or $\rho^2 \sin(\phi)$ when going to spherical coordinates). The integral in c) becomes then $\int \int_R u^4/v^4 du dv$. The region R is the triangle bounded by the edges $(0, 0), (1, 1), (-1, 1)$.

Problem 8) (10 points)

- a) (3 points) Find all the critical points of the function $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$.
- b) (3 points) Classify the critical points.
- c) (2 points) Locate the local and absolute maxima of f .
- d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy -plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = (t, \sin t, t^2 \cos t)$ for $0 \leq t \leq \pi$. Let $f(x, y, z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Problem 11) (10 points)

A cylindrical building $x^2 + (y - 1)^2 = 1$ is intersected with the paraboloid $z = 4 - x^2 - y^2$.

- a) Parametrize the intersection curve and set up an integral for its arc length.
- b) Find a parametrization of the surface obtained by intersecting the paraboloid with the solid cylinder $x^2 + (y - 1)^2 \leq 4$ and set up an integral for its surface area.

SECTION SPECIFIC PROBLEMS: PROBLEMS FOR REGULAR AND PHYSICS SECTIONS:

Problem 12A) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x, y) = (y^2, x^2)$ in the clockwise direction around the triangle in the xy -plane defined by the points $(0, 0)$, $(1, 0)$ and $(1, 1)$ in two ways:

- a) (5 points) by evaluating the three line integrals.
- b) (5 points) using Greens theorem.

Problem 13A) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x, y, z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle$ with $t \in [0, 2\pi]$.

Problem 14A) (10 points)

Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy -plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int \int_S \vec{F} \cdot d\vec{S}$ of the vectorfield

$$\vec{F} = \left\langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \right\rangle$$

through S using the divergence theorem.

PROBLEMS FOR BIO CHEM SECTIONS.

Problem 12B) (10 points)

At a county fair, two competing booths offer different games. Each charges the same price to play. At booth A, you toss a loaded coin. If you get heads for the first time on the n 'th toss, you win n dollars. Let the random variable X_A be the pay-off from this game. At booth B, a trained monkey picks a point on the interval $[0, 2]$. (The monkey is honest, so the points it picks are uniformly distributed.) If it picks the point x , you win x^2 . Let the random variable X_B be the payoff from this game.

Find the probability densities $p_{X_A}(x)$ and $p_{X_B}(x)$ and their expected values. Which game has a higher expected value?

Problem 13B) (10 points)

King-Kong flips a biased coin that lands heads 70% of the time. He makes 80 flips.

- a) What are the expected number of heads?
- b) Give an exact expression for the probability that there are 62 or more heads in this experiment. You don't have to compute the numerical value.

Problem 14B) (10 points)

Consider an experiment which consists of throwing three dice, each with sides numbered 1-6.

- a) What is the probability of getting more than 2 on at least one of the dice?
- b) What is the probability that the sum showing on all three dice is less than or equal to 4.
- c) Given that the first die shows 3, what is the probability that the sum of all three dice is 7?