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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.
- Please write neatly.
- Do not use notes, books, calculators, computers, or other electronic aids.
- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.
- You have 180 minutes time to complete your work.
- The iochem section can ignore problems with vector fields and line integrals.

MWF 10 Samik Basu
MWF 10 Joachim Krieger
MWF 11 Matt Leingang
MWF 11 Veronique Godin
TTH 10 Oliver Knill
TTH 115 Thomas Lam

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10

12A		10
13A		10
14A		10

12B		10
13B		10
14B		10

Total:		140
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Problem 1) True/False questions (20 points)

- 1)  T  F The distance from  $(1, 2, -1)$  to  $(3, -2, 1)$  is  $(-2, 4, -2)$ .
- 2)  T  F The plane  $y = 3$  is perpendicular to the  $xz$  plane.
- 3)  T  F All functions  $u(x, y)$  that obey  $u_x = u$  at all points obey  $u_y = 0$  at all points.
- 4)  T  F The best linear approximation at  $(1, 1, 1)$  to the function  $f(x, y, z) = x^3 + y^3 + z^3$  is the function  $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$
- 5)  T  F If  $f(x, y)$  is any function of two variables, then  $\int_0^1 \left( \int_x^1 f(x, y) dy \right) dx = \int_0^1 \left( \int_y^1 f(x, y) dx \right) dy$ .
- 6)  T  F Let  $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$  be the unit circle in the plane and  $\vec{F}(x, y)$  a vector field satisfying  $|\vec{F}| \leq 1$ . Then  $-2\pi \leq \int_C \vec{F} \cdot d\vec{r} \leq 2\pi$ .
- 7)  T  F Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors. Then the vectors  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  always point in different directions.
- 8)  T  F If all the second-order partial derivatives of  $f(x, y)$  vanish at  $(x_0, y_0)$  then  $(x_0, y_0)$  is a critical point of  $f$ .
- 9)  T  F If  $\vec{a}, \vec{b}$  are vectors, then  $|\vec{a} \times \vec{b}|$  is the area of the parallelogram determined by  $\vec{a}$  and  $\vec{b}$ .
- 10)  T  F The distance between two points  $A, B$  in space is the length of the curve  $\vec{r}(t) = A + t(B - A)$ ,  $t \in [0, 1]$ .
- 11)  T  F The function  $f(x, y) = xy$  has no critical point.
- 12)  T  F The length of a curve does not depend on the chosen parameterization.
- 13)  T  F The equation  $\rho = 1$  in spherical coordinates defines a cylinder.
- 14)  T  F For any numbers  $a, b$  satisfying  $|a| \neq |b|$ , the vector  $\langle a - b, a + b \rangle$  is perpendicular to  $\langle a + b, b - a \rangle$ .
- 15)  T  F The line integral of  $\vec{F}(x, y) = \langle -y, x \rangle$  along the counterclockwise oriented boundary of a region  $R$  is twice the area of  $R$ .
- 16)  T  F There is no surface for which both the parabola and the hyperbola appear as traces.
- 17)  T  F If  $(u, v) \mapsto \vec{r}(u, v)$  is a parameterization for a surface, then  $\vec{r}_u(u, v) + \vec{r}_v(u, v)$  is a vector which lies in the tangent plane to the surface.
- 18)  T  F When using spherical coordinates in a triple integral, one needs to include the volume element  $dV = \rho^2 \cos(\phi) d\rho d\phi d\theta$ .

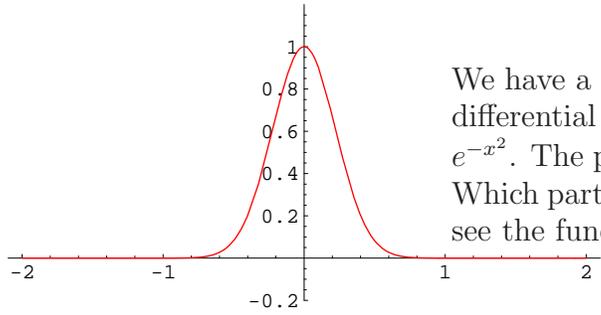
**TF PROBLEMS FOR REGULAR AND PHYSICS SECTIONS:**

- 19)  T  F A connected surface in space for which all normal vectors are parallel to each other must be part of a plane.
- 20)  T  F A vector field  $\vec{F} = \langle P(x, y), Q(x, y) \rangle$  is conservative in the plane if and only if  $P_y(x, y) = Q_x(x, y)$  for all points  $(x, y)$ .

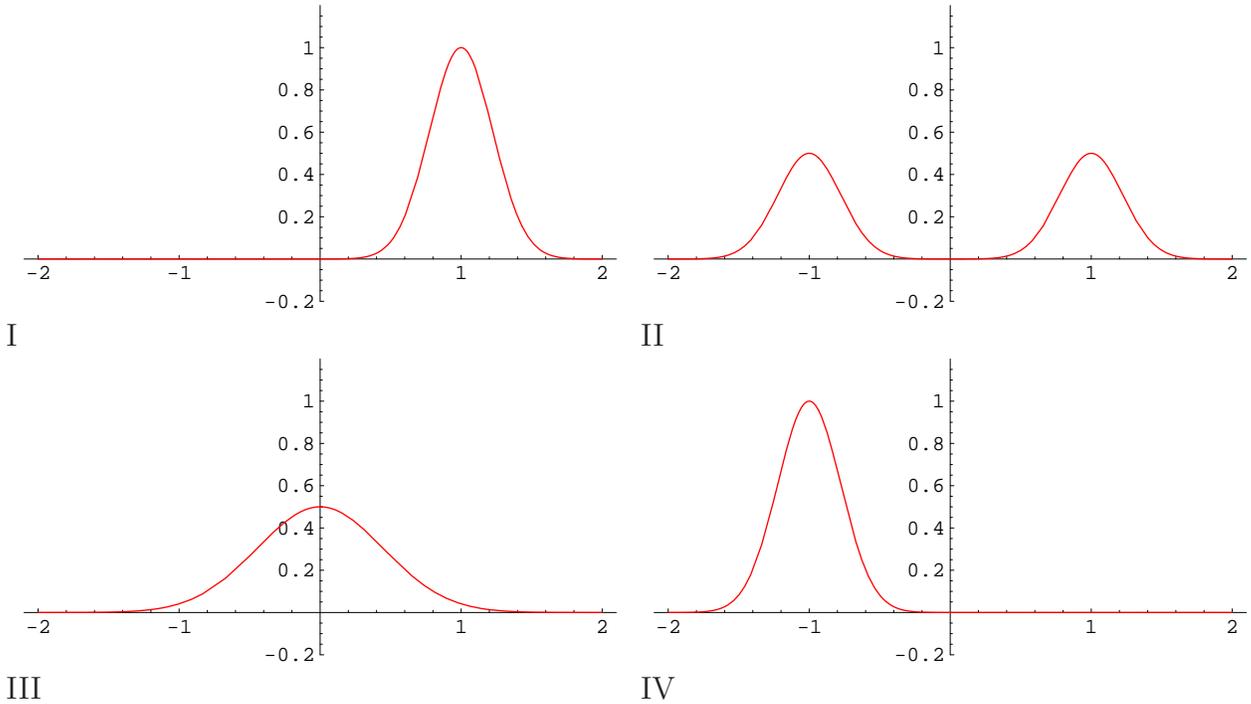
TF PROBLEMS FOR BIOCHEM SECTIONS:

- 21)  T  F      Suppose  $X$  and  $Y$  are two random variables such that  $E[X] > E[Y]$ . Is it always the case that  $P[X > Y] > 1/2$ ?
- 22)  T  F      If  $\phi$  is the density function of a random variable  $\chi$ , then  $\int \phi(x) dx$  is the expectation  $E\chi$  of the random variable.

Problem 2) (10 points)



We have a function  $u(t, x)$  which is a solution to partial differential equation. In all cases, we have  $u(0, x) = e^{-x^2}$ . The picture to the left shows this function  $u(0, x)$ . Which partial differential equation is involved, when you see the function  $u(1, x)$  as a graph?



Enter I,II,III,IV here	Equation
	$u_t(x, t) = u_x(x, t)$
	$u_t(x, t) = u_{xx}(x, t)$
	$u_{tt}(x, t) = u_{xx}(x, t)$
	$u_t(x, t) = -u_x(x, t)$

Problem 3) (10 points)

- a) Find an equation for the plane  $\Sigma$  passing through the points  $P = (1, 0, 1)$ ,  $Q = (2, 1, 3)$  and  $R = (0, 1, 5)$ .
- b) Find the distance from the origin  $O = (0, 0, 0)$  to  $\Sigma$ .
- c) Find the distance from the point  $P$  to the line through  $Q, R$ .
- d) Find the volume of the parallelepiped with vertices  $O, P, Q, R$ .

Problem 4) (10 points)

The equation  $f(x, y, z) = e^{xyz} + z = 1 + e$  implicitly defines  $z$  as a function  $z = g(x, y)$  of  $x$  and  $y$ .

- a) Find formulas (in terms of  $x, y$  and  $z$ ) for  $g_x(x, y)$  and  $g_y(x, y)$ .
- b) Estimate  $g(1.01, 0.99)$  using linear approximation.

Problem 5) (10 points)

Find the surface area of the surface  $S$  parametrized by  $\vec{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$  for  $(u, v)$  in the disc  $D = \{u^2 + v^2 \leq 1\}$ .

Problem 6) (10 points)

Find the local and global extrema of the function  $f(x, y) = x^3/3 + y^3/3 - x^2/2 - y^2/2 + 1$  on the disc  $\{x^2 + y^2 \leq 4\}$ .

- a) Classify every critical point inside the disc  $x^2 + y^2 < 4$ .
- b) Find the extrema on the boundary  $\{x^2 + y^2 = 4\}$  using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of  $D$ .

Problem 7) (10 points)

a) Given two nonzero vectors  $\vec{u} = \langle a, b, c \rangle$  and  $\vec{v} = \langle d, e, f \rangle$  in  $\mathbf{R}^3$ , write down a formula for the cosine of the angle between them. Find a nonzero vector  $\vec{v}$  that is perpendicular to  $\vec{u} = \langle 3, 2, 1 \rangle$ . Describe geometrically the set of all  $\vec{v}$ , including zero, that are perpendicular to this vector  $\vec{u}$ .

b) Consider a function  $f$  of three variables. Explain with a picture and a sentence what it means geometrically that  $\nabla f(P)$  is perpendicular to the level set of  $f$  through  $P$ .

c) Assume the gradient of  $f$  at  $P$  is nonzero. Write a few sentences that would convince a skeptic that  $\nabla f(P)$  is perpendicular to the level set of  $f$  at the point  $P$ .

d) Assume the level set of  $f$  is the graph of a function  $g(x, y)$ . Explain the relation between the gradient of  $g$  and the gradient of  $f$ . Especially, how do you relate the orthogonality of  $\nabla f$  to the level set of  $f$  with the orthogonality of  $\nabla g$  to the level set of  $g$ ?

Problem 8) (10 points)

Let  $R$  be the region inside the circle  $x^2 + y^2 = 4$  and above the line  $y = \sqrt{3}$ . Evaluate

$$\iint_R \frac{y}{x^2 + y^2} dA.$$

Problem 9) (10 points)

A region  $W$  in  $\mathbf{R}^3$  is given by the relations

$$\begin{aligned}x^2 + y^2 &\leq z^2 \leq 3(x^2 + y^2) \\ 1 &\leq x^2 + y^2 + z^2 \leq 4 \\ x &\geq 0\end{aligned}$$

1. Sketch the region  $W$ .
2. Find the volume of the region  $W$ .

Problem 10) (10 points)

Consider the vector field

$$\vec{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

defined everywhere in the plane  $\mathbf{R}^2$  except at the origin.

a) Let  $C$  be any closed curve which bounds a region  $D$ . Assume that  $(0, 0)$  is not contained in  $D$  and does not lie on  $C$ . Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0 .$$

b) Let  $C$  be the unit circle oriented counterclockwise. What is  $\int_C \vec{F} \cdot d\vec{r}$ ? Explain why your answer shows that there is no function  $f$  for which  $\vec{F}(x, y) = \nabla f(x, y)$  everywhere except at the origin  $(0, 0)$ .

Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function  $f(x, y, z) = xyz$  over the solid in space described by the inequalities  $0 \leq z \leq \sqrt{(1 - x^2 - y^2)}, x^2 + y^2 \leq 1, x - y \geq 0, y \geq 0$ .

**SECTION SPECIFIC PROBLEMS FOR REGULAR AND PHYSICS SECTIONS.**

Problem 12A) (10 points)

Let  $\vec{F}(x, y)$  be a vector field in the plane given by the formula

$$\vec{F}(x, y) = \left\langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4 + 1}} \right\rangle .$$

If  $C$  is the path which goes from from  $(-1, 0)$  to  $(1, 0)$  along the semi circle  $x^2 + y^2 = 1, y \geq 0$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$ .

Problem 13A) (10 points)

In appropriate units, the charge density  $\sigma(x, y, z)$  in a region in space is given by  $\sigma = \nabla \cdot \vec{E} = \text{div}(\vec{E})$ , where  $\vec{E}$  is the electric field. Consider the cube of side lengths 1 given by  $0 \leq x, y, z \leq 1$ . What is the total charge in this cube if

$$\vec{E} = \langle x(1-x) \log(1+xyz), y(1-y) \tan(x^3 + y^3 + z^3), z(1-z)e^{\sqrt{x+y}} \rangle.$$

(The total charge is the integral of the charge density over the cube.)

Problem 14A) (10 points)

a) By calculating the integral  $\int_S \vec{F} \cdot d\vec{S}$  directly, find the flux of the vector field  $\vec{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through the sphere  $x^2 + y^2 + z^2 = 9$ , where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field  $\mathbf{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through the sphere  $x^2 + y^2 + z^2 = 9$  using the divergence theorem.

c) Explain **in words** without invoking any integral theorem, why the flux integral of the vector field  $\vec{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through any sphere with positive radius centered at  $(0, 0, 0)$  is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

**FOR BIOCHEM SECTIONS.**

Problem 12B) (10 points)

It is known that spark plugs produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the spark plugs in packages of ten. The company offers a refund if more than one of the screws is defective. What proportion of the packages sold must the company replace?

Problem 13B) (10 points)

The monthly worldwide average number of cases of a certain disease is 3. What is the probability that there will be at least two such cases in the next month?

Hint. The probability distribution function we use is the geometric distribution  $f(x) = \lambda^x \frac{e^{-\lambda}}{x!}$  with  $\lambda = 3$ .

Problem 14B) (10 points)

From a group of 10 women and 15 men, we wish to form a committee consisting of 4 women and 6 men. There are three men who cannot serve together in any committee. How many different committees can be formed?