5/23/2006, FINAL EXAM **Math 21a, Spring 2006**

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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.
- Please write neatly.
- Do not use notes, books, calculators, computers, or other electronic aids.
- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10

11A		10
12A		10
13A		10

11B		10
12B		10
13B		10

Total:		140

Problem 1) True/False questions (20 points)

1) T F For any two vectors $\vec{v}$ and $\vec{w}$ one has $\text{proj}_{\vec{v}}(\vec{v} \times \vec{w}) = \vec{0}$.

2) T F Any parameterized surface S is either the graph of a function $f(x, y)$ or a surface of revolution.

3) T F If the directional derivative $D_{\vec{v}}(f)$ of f into the direction of a unit vector $\vec{v}$ is zero, then $\vec{v}$ is perpendicular to the level curve of f.

4) T F The linearization $L(x, y)$ of $f(x, y) = 5x - 100y$ at $(0, 0)$ satisfies $L(x, y) = 5x - 100y$.

5) T F If a parameterized curve $\vec{r}(t)$ intersects a surface $\{f = c\}$ at a right angle, then at the point of intersection we have $\nabla f(\vec{r}(t)) \times \vec{r}\,'(t) = 0$.

6) T F The curvature of the curve $\vec{r}(t) = \langle \cos(3t^2), \sin(6t^2) \rangle$ at the point $\vec{r}(1)$ is larger than the curvature of the curve $\vec{r}(t) = \langle 2\cos(3t), 2\sin(6t) \rangle$ at the point $\vec{r}(1)$.

7) T F At every point (x, y, z) on the hyperboloid $x^2 - y^2 + z^2 = 10$, the vector $\langle x, -y, z \rangle$ is normal to the hyperboloid.

8) T F The set $\{\phi = \pi/2, \theta = \pi/2\}$ in spherical coordinates is the positive y-axis.

9) T F The integral $\int_0^1 \int_0^{2\pi} r^2 \sin(\theta) \;\; d\theta \; dr$ is equal to the area of the unit disk.

10) T F If three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ attached at the origin are in a common plane, then $\vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w}) = 0$.

11) T F If a function $f(x, y)$ has a local minimum at $(0, 0)$, then the discriminant D must be positive.

12) T F The integral $\int_0^1 \int_y^1 f(x, y) \; dy \; dx$ represents a double integral over a bounded region in the plane.

13) T F The following identity is true: $\int_0^3 \int_0^x x^2 \; dydx = \int_0^3 \int_y^3 x^2 \; dx \; dy$

14) T F There is a quadric, for which all three traces are hyperbola.

15) T F The curvature of a space curve $\vec{r}(t)$ is a vector perpendicular to the acceleration vector $\vec{r}\,''(t)$.

TF problems 16-20 are for regular sections only:

16) T F Assume S is the unit sphere oriented so that the normal vector points outside. Let S^+ be the upper hemisphere and S^- the lower hemisphere. If a vector field $\vec{F}$ has divergence zero, then the flux of $\vec{F}$ through S^+ is equal to the flux of $\vec{F}$ through S^-.

17) T F If a vector field $\vec{F}$ is defined at all points in three-space except at the origin and curl(F̃) = $\tilde{0}$ everywhere, then the line integral of $\vec{F}$ around any closed path not passing through the origin is zero.

18) T F Every vector field which satisfies $\text{curl}(\vec{F}) = 0$ everywhere in space can be written as $\vec{F} = \text{grad}(f)$ for some function f.

19) T F Let $\vec{F}$ be a vector field and let S be an oriented surface $\vec{r}(u, v)$. Then $20 \int \int_S \vec{F} \cdot dS = \int \int_S \vec{G} \cdot dS$, where $\vec{G} = 20\vec{F}$.

20) T F Consider the surface S given by the equation $z^2 = f(x, y)$. If $(x, y, z) = (x, y, \sqrt{f(x, y)})$ is a point on the surface with maximal distance from the origin, it is a local maximum of $g(x, y) = x^2 + y^2 + f(x, y)$.

TF problems 21-25 are for biochem sections only:

21) T F If the sum of the variances of X and Y is equal to the variance of $X + Y$, then the random variables X and Y are independent.

22) T F Let X be a random variable with zero mean $E(X) = 0$. Then the expectation $E(X^2)$ is equal to the variance $\text{D}(X)$.

23) T F If A, B are two events which have positive probability and $P(A|B)$ as well as $P(B|A)$ are known, then we can compute $P(A)/P(B)$.

24) T F The function $f(x) = e^{-x}$ on $[0, \infty)$ is the distribution function of a random variable.

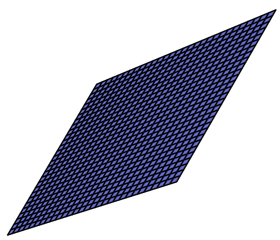
25) T F Suppose you throw two fair coins n times. The probability to have k times head is $k!(n - k)!/2^n$.

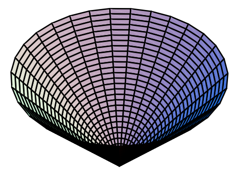
Problem 2) (10 points)

Match the parameterized surface formulas and pictures with the formulas for the implicit surfaces. No justifications are needed.

A)	$\vec{r}(u,v) = \langle 1+u, v, u+v \rangle$
B)	$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v \rangle$
C)	$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$
D)	$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$
E)	$\vec{r}(u,v) = \langle v, \sin(u), \cos(u) \rangle$

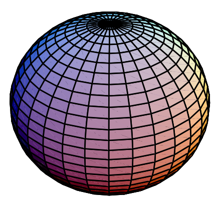
I



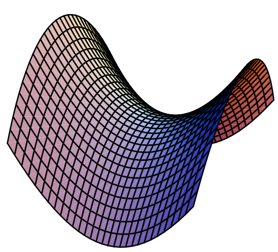


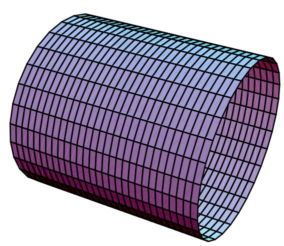
II

III



IV





V

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
		$y^2 + z^2 = 1$
		$x + y - z = 1$
		$x^2 + y^2 + z^2 = 1$
		$x^2 + y^2 - z^2 = 0$
		$x^2 - y^2 - z = 0$

Problem 3) (10 points)

Match the formulas and theorems with their names. No justifications are needed.

N) $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$
U) $f_{xy}(x,y) = f_{yx}(y,x)$
M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$
E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$
R) $\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$
L) $(\vec{u}\times\vec{v})\cdot\vec{w}$
F) $|\vec{v}\cdot\vec{w}| \leq |\vec{v}|\cdot|\vec{w}|$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$
G) $|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
	Fubini theorem
	Clairot theorem
	vector projection
	scalar projection
	chain rule
	dot product formula
	scalar triple product
	Cauchy-Schwarz inequality
	Pythagorean theorem
	Triangle inequality

Problem 4) (10 points)

Let L be the line $\vec{r}(t) = \langle t, 0, 0\rangle$. We are also given a point $Q = (3, 3, 0)$ in space.

a) (2 points) What is the distance $d((x,y,z), Q)$ between a general point (x,y,z) and Q?

b) (3 points) What is the distance $d((x,y,z), L)$ between the point (x,y,z) and the line L?

c) (3 points) Find the equation for the set C of all points (x,y,z) satisfying

$$d((x,y,z),Q) = d((x,y,z),L)\ .$$

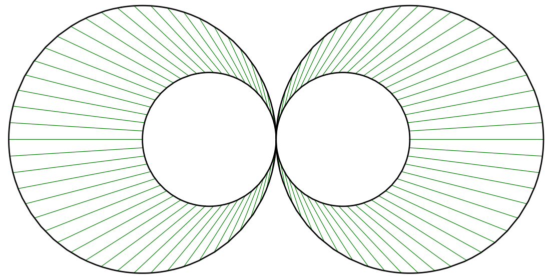
d) (2 points) Identify the surface.

Problem 5) (10 points)

Find the area of the region in the plane given in polar coordinates by

$$\{(r,\theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi\ \}\ .$$

The region is the shaded part in the figure.

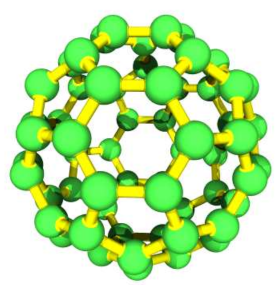


Problem 6) (10 points)

A microscopic bucky ball C60 is located on a gold surface. The surface produces the electric potential $f(x,y) = x^4 + y^4 - 2x^2 - 8y^2 + 5$.

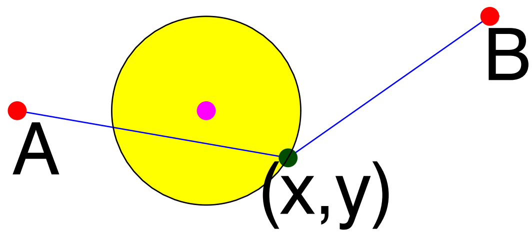
a) (7 points) Find all critical points of f and classify them.

b) (3 points) The fullerene will settle at a global minimum of $f(x,y)$. Find the global minima of the function $f(x,y)$.



Problem 7) (10 points)

A circular wheel with boundary $g(x,y) = x^2 + y^2 = 1$ has the boundary point (x,y) connected to two points $A = (-2,0)$ and $B = (3,1)$ by rubber bands. The potential energy at position (x, y) is by Hooks law equal to $f(x, y) = (x+2)^2 + y^2 + (x-3)^2 + (y-1)^2$, the sum of the squares of the distances to A and B. Our goal is to find the position (x, y) for which the energy is minimal. To find this position for which the wheel is at rest, minimize $f(x, y)$ under the constraint $g(x, y) = 1$.



Problem 8) (10 points)

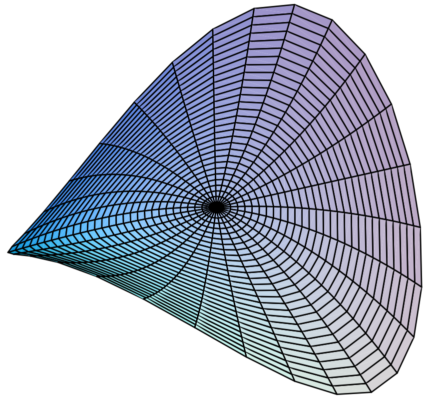
a) (5 points) Find the surface area of the parameterized surface

$$\vec{r}(u, v) = \langle u - v, u + v, uv \rangle$$

with $u^2 + v^2 \leq 1$.

b) (3 points) Find an implicit equation $g(x, y, z) = 0$ for this surface.
Hint: Look at $y^2 - x^2$.

c) (2 points) What is the name of the surface?



Problem 9) (10 points)

a) (4 points) Find the tangent plane to the surface $f(x, y, z) = zx^5 + y^5 - z^5 = 1$ at the point $(1, 1, 1)$.

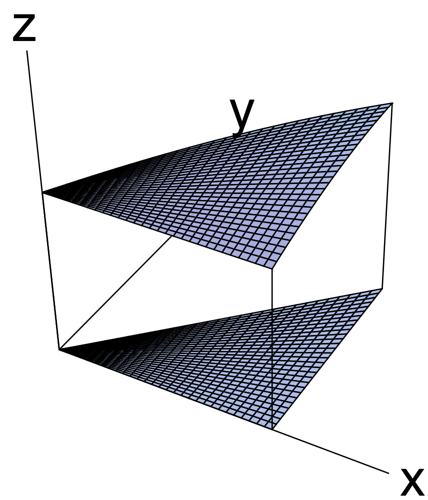
b) (3 points) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at at the point $(1, 1, 1)$.

c) (3 points) Near the point $(1, 1, 1)$, the surface can be written as a graph $z = g(x, y)$. Find the partial derivative $g_x(1, 1)$.

Problem 10) (10 points)

A tower E with base $0 \leq x \leq 1, 0 \leq y \leq x$ has a roof $f(x, y) = \sin(1 - y)/(1 - y)$. Find the volume of this solid. The solid is given in formulas by

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq f(x, y) \} .$$



The following problem 11 A is for regular sections only:

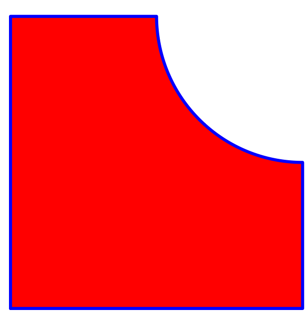
Problem 11 A) (10 points)

Let $\vec{F} = \langle y, 2x + \tan(\tan(y)) \rangle$ be a vector field in the plane and let C be the boundary of the region

$$G = \{0 \leq x \leq 2, 0 \leq y \leq 2, (x-2)^2 + (y-2)^2 \geq 1\}$$

oriented counter clock-wise. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r} \; .$$

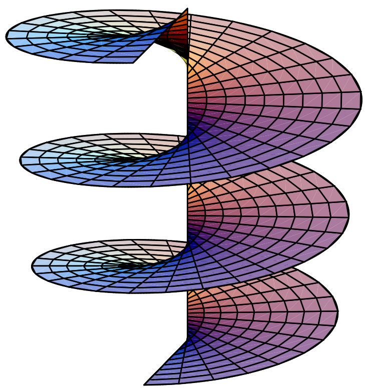


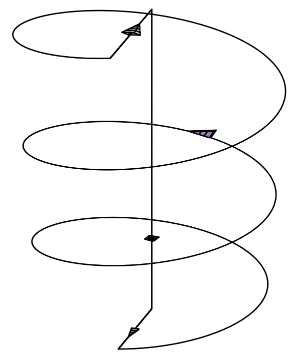
The following problem 12 A is for regular sections only:

Problem 12 A) (10 points)

Let S be the surface of a turbine blade parameterized by $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), t\rangle$ for $t \in [0, 6\pi]$ and $s \in [0,1]$. Let $\vec{F} = \text{curl}(\vec{G})$ denote the velocity field of the water velocity, where $\vec{G}(x,y,z) = \langle -y + (x^2+y^2-1), x + (x^2+y^2-1), (x^2+y^2-1) \rangle$. Compute the power of the turbine which is given by the flux of $\vec{F} = \text{curl}(\vec{G})$ through S.

Hint. The boundary C of the surface S consists of 4 paths:
$\vec{r_1}(t) = \langle \cos(t), \sin(t), t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_2}(s) = \langle 1 - s, 0, 6\pi \rangle$, $s \in [0, 1]$.
$\vec{r_3}(t) = \langle 0, 0, 6\pi - t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_4}(s) = \langle s, 0, 0 \rangle$, $s \in [0, 1]$.





The following problem 13 A is for regular sections only:

Problem 13 A) (10 points)

Let $\vec{F}(x,y,z) = \langle z^2, -z^5 + z\sin(e^{\sin(x)}), (x^2+y^2) \rangle$. Let S denote the part of the graph $z = 9 - x^2 - y^2$ lying above the xy-plane oriented so that the normal vector points upwards. Find the flux of $\vec{F}$ through the surface S.

Hint. You might also want to look at the surface $D = \{x^2 + y^2 \le 9, z = 0\}$ lying in the xy-plane.

The following problem 11 B is for biochem sections only:

Problem 11 B) (10 points)

A certain state has license plates with 6 entries. These entries can be letters (26 of them) and digits (10 or them). How many license plates are there under the following restrictions?

a) (3 points) Assume that we allow any combination of letters and digits.

b) (3 points) Assume that the license plates start with three letters and then have three digits.

c) (4 points) Assume, as in a), that we allow any combination of letters and digits. Assume that all plates are as likely. What are the odds of getting four or more letters on your plate?

The following problem 12 B is for biochem sections only:

Problem 12 B) (10 points)

We have 10 coins. Eight of them are fair. Two coins are not fair: one coin has two tails and one has two heads. Say I pick a coin randomly from these ten coins and that I throw this coin six times.

a) (4 points) What is the probability that the chosen coin is the one with two tails if I obtained six tails?

b) (4 points) Are the following events independent? My first throw turns out to be a tail. My second throw is a head.

c) (2 points) Explain to a non-mathematician what your answer in b) means and why you would expect it.

The following problem 13 B is for biochem sections only:

Problem 13 B) (10 points)

Taxis and buses travel from Central Square to Harvard square. Their travel time in minutes are independent random variables T and B with probability densities

$$p_T(t) = \begin{cases} 2e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad p_B(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0. \end{cases} .$$

a) (3 points) Find the expected travel time to Harvard square both in a taxi and in a bus.
b) (3 points) What is the probability that a bus trip to Harvard square takes between 5 and 10 minutes?
c) (4 points) Peter is at Harvard square. Andrew and Kevin have a message to give to him. They leave Central square at the same time, but Andrew takes a bus and Kevin a taxi. Let Y be the number of minutes before Peter gets the message. Hence Y is the time before the **first** one of Andrew and Kevin arrives in Harvard square. What is the probability density of the random variable Y?