Math 21a Hourly 2

(F 2000)

1) __ 2) __ 3) __ 4) __ 5) __ 6) __ : Total ______

Name: ____

Circle the name of your Section TA:

Allcock • Chen • Karigiannis • Knill • Liu • Rasmussen • Rogers • Taubes • D. Winter • R. Winters

Instructions:

- Print your name in the line above and circle the name of your section TF.
- Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page or the extra blank pages at the end of the exam booklet. Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- You have 2 hours to complete your work.
- Problems 1-6 count 8 points each.
- Under no circumstances should you unstaple this exam booklet or detach any of the pages as the graders will not take responsibility for lost pages from exam booklets.

In agreeing to take this exam, you are implicitly accepting Harvard University's honor code.

1. Suppose that f(x, y) is a function with the following properties:

   - $f(1, 2) = 6$.
   - The directional derivative at (1, 2) of f in the direction of $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$ is 5
   - The directional derivative at (1, 2) of f in the direction of $\mathbf{v} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is $\frac{1}{\sqrt{2}}$.

   Given this information, answer the following questions:

   a) Write the gradient of f at (1, 2).
   b) Write an equation for the tangent plane to the graph of f at the point (1, 2, 6).
   c) Estimate the value of f(1.02, 2.05).

2. Let a, b and c be non-negative constants and F the function

$$F(x, y, z) = - x \ln(x) - y \ln(y) - z \ln(z) - a x - b y - c z.$$

Find the stationary points of F where $x > 0$, $y > 0$, $z > 0$ and $x + y + z = 1$. Don't search for stationary points where any of x, y or z are zero.

(If you are interested in where this question comes from, here is the answer: This sort of extremal problem arises when predicting the shapes of biological molecules. For example, proteins are complicated molecules that often fold into a number of stable shapes. Moreover, the biological effect of the protein is very sensitive to the particulars of its shape. If a certain protein has three possible folded shapes which we label as X, Y and Z, then in a given environment, a fraction, x of the protein molecules will have shape X, a fraction y will have shape Y and a fraction z will have shape Z. It is a fundamental problem in biochemistry to predict these fractions x, y and z. In particular, these fractions are known to depend on the energies, a, b and c of the shapes X, Y and Z; and the correct values for x, y and z are the constrained minima of $F(x, y, z) = -x \ln(x) - y \ln(y) - z \ln(z) - a x - b y - c z$ where the constraint is $x + y + z = 1$. In this context, the function F is called the *free energy*.)

3. Integrate the function $f(x, y) = x$ over the interior of the region where $x \geq 0$ and $x^2/4 + y^2 \leq 1$.

4. Let $f(x, y) = x^4 - 2x^2 + 3y^3 - 9y$.
   a) Find all stationary points of f.
   b) Classify each of the points found in Part a as local maxima, minima or saddle.
   c) Find all points on the level set $f = 2$ where the tangent line is parallel to the y axis.

5. Evaluate $\int_0^1 \left(\int_0^{\mathrm{Arc\,sin}(x)} dy \right) dx$.

6. A rectangular, open top aquarium with a slate bottom and glass side is to hold 20,000,000 cubic centimeters of water. Suppose that glass cost 10 cents per square centimeter and slate cost 50 cents per square centimeter. Find the dimensions (height, length and width) which minimize the price of the aquarium. Please justify your answer.