Name: ______________________________

Math 21a Midterm 1 Wednesday, October 22nd 2003

Please circle your section:

Namhoon Kim Mauro Bruanstein MWF 9–10	Teru Yoshida Nathan Lange MWF 10–11	Tom Coates Barbara Richter MWF 10–11	Tom Coates Darin Ragozzine MWF 11–12
Jonathan Kaplan Chris Phillips MWF 11–12	Vivek Mohta Kory Byrns MWF 11–12		Vivek Mohta Ann Lai MWF 12–1
Abhinav Kumar Justin Ho Tu/Th 10–11:30	Sabin Cautis Shih En Lu Tu/Th 10–11:30		Andy Engelward Brad Burns T/Th 11:30–1

Question	Points	Score
1	18	
2	11	
3	17	
4	12	
5	16	
6	14	
7	12	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in order – move on to another problem if you find that you're stuck and are spending too much time on one problem.

To receive full credit on a problem, you need to justify your answers carefully (unless the question specifically says otherwise). Unsubstantiated answers will receive little or no credit! Please be sure to write neatly – illegible answers will also receive little or no credit. If you're asked to provide a sketch of something, do so as neatly as possible – you could lose points if your sketch is difficult to understand.

If you need more space for a problem then use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

Please note that you are not allowed to use any notes or calculators during this test.

Good luck! Focus and do well!

Question 1. (18 points total)

Consider the plane $x + 3z = 1$, and the line parametrized by $x = 2t + 1$, $y = 4t - 1$, $z = 3t$

(a) (4 points) Find two different unit vectors that are normal to the plane.

$\langle 1, 0, 3 \rangle$ is normal to plane $1 \cdot x + 0 \cdot y + 3z = 1$, so is $\langle -1, 0, -3 \rangle$. So two unit vectors are

$$\frac{\langle 1,0,3 \rangle}{|\langle 1,0,3 \rangle|} = \left\langle \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \right\rangle \text{ and } \left\langle \frac{-1}{\sqrt{10}}, 0, \frac{-3}{\sqrt{10}} \right\rangle$$

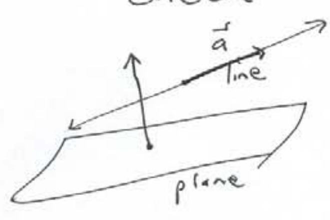
(b) (2 points) Find a vector **a** that is parallel to the line.

line is $\langle 2t+1, 4t-1, 3t \rangle = \langle 1, -1, 0 \rangle + t\langle 2, 4, 3 \rangle$

$\vec{a} = \langle 2, 4, 3 \rangle$ (or any multiple of $\vec{a}$)

(c) (3 points) Explain why the line is not parallel to the plane.

check $\vec{a} \cdot \vec{n}$. If the line is parallel, then $\vec{a}$ will be $\perp$ to $\vec{n}$,



but $\vec{a} \cdot \vec{n} = \langle 2,4,3 \rangle \cdot \langle 1,0,3 \rangle$

$= 11 \neq 0$,

so $\vec{a}$ is not $\perp$ to $\vec{n}$,

and so the line is not parallel to the plane

Question 1 continued

(Continue working with plane $x+3z=1$, and the line parametrized by $x=2t+1, \quad y=4t-1, \quad z=3t$)

(d) (3 points) Find the point of intersection of the line and the plane.

plane: $x+3z=1$, so when does $\langle 2t+1, 4t-1, 3t\rangle$ satisfy $x+3z=1$? (then that point will be on both the line and the plane)

so $2t+1+3(3t)=1 \ldots 11t+1=1$, when $t=0$,

so $\langle 2\cdot 0+1, 4\cdot 0-1, 3\cdot 0\rangle = \langle 1,-1,0\rangle$ is the point

(e) (3 points) Find an equation for a plane that contains both the line and the origin.

Since both $\langle 1,-1,0\rangle$ and $\langle 0,0,0\rangle$ are on this plane, then the vector $\langle 1,-1,0\rangle - \langle 0,0,0\rangle = \langle 1,-1,0\rangle$ is parallel to the plane, as is $\vec{a}=\langle 2,4,3\rangle$, the direction vector given for the line in part (b).

So a possible normal vector $n=\langle 1,-1,0\rangle \times \langle 2,4,3\rangle$

$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 2 & 4 & 3 \end{vmatrix} = \langle -3,-3,6\rangle$. Can take $\vec{n}=\langle 1,1,-2\rangle$.

then $\vec{n}\cdot(\langle x,y,z\rangle - \langle 0,0,0\rangle)=0$ is ~~the~~ an equation for the plane

so $\langle 1,1,-2\rangle\cdot\langle x,y,z\rangle = x+y-2z=0$

(f) (3 points) Find the angle between the plane in part (e) and the original plane, $x+3z=1$ (recall that the angle between two planes is defined as the acute angle between their normal vectors). If you weren't able to find the plane in part (e), then use the plane $-2x+7y-12z=0$ instead.

normal for the new plane $\vec{n}_1=\langle 1,1,-2\rangle$,

normal for original plane $\vec{n}_2=\langle 1,0,3\rangle$,

so $\vec{n}_1\cdot\vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos\theta \rightarrow$ angle between planes

$\cos\theta = \dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{-5}{\sqrt{6}\sqrt{10}}$ so $\theta=\cos^{-1}\left(\dfrac{-5}{\sqrt{60}}\right)$

but we want the <u>acute</u> angle, so the angle between the planes is $\cos^{-1}\left(\dfrac{5}{\sqrt{60}}\right)$

Question 2. (11 points total) Let **a** and **b** be two vectors such that $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(a) (3 points) What is $(-3\mathbf{b}) \times (4\mathbf{a})$?

$$= -12(\vec{b} \times \vec{a}) = 12(\vec{a} \times \vec{b}) = 24\vec{i} + 24\vec{j} - 36\vec{k}$$

(b) (2 points) Find $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b})$ if it exists. If it doesn't exist then explain why not.

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ (and $\vec{b}$) so $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

(c) (2 points) Find $\mathbf{a} \times (\mathbf{a} \bullet \mathbf{b})$ if it exists. If it doesn't exist then explain why not.

Can't do it: $\vec{a} \cdot \vec{b}$ is a scalar, not a vector, so $\vec{a} \times (\vec{a} \cdot \vec{b})$ is not defined.

(d) (4 points) Suppose that not only does **a** x **b** = 2**i** + 2**j** - 3**k**, but also that |**a** x **b**| = |**a**| |**b**|. Which of the following statements *must* be true? (simply circle all the statements that must be true – no explanations are necessary).

(i) $\mathbf{a} \neq \mathbf{b}$

(ii) **a** and **b** are parallel

(iii) **a** is perpendicular to **b**

(iv) either **a** or **b** must be a unit vector.

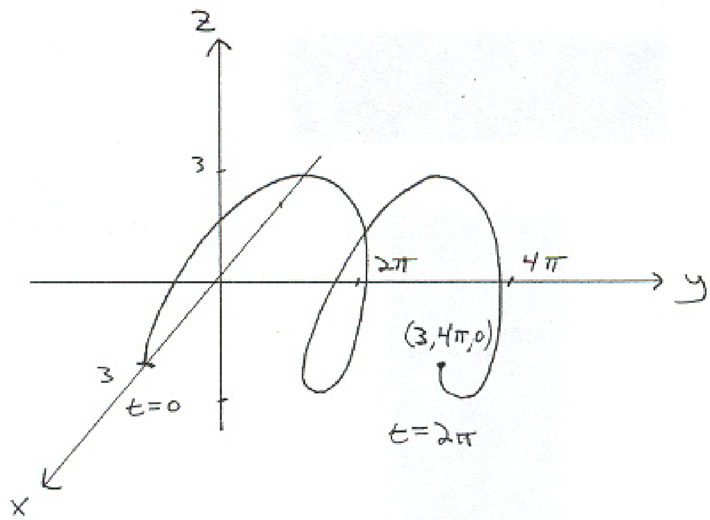
Since $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ (θ angle btw $\vec{a}, \vec{b}$)
then $\sin\theta = 1$, so $\theta = 90^\circ$, so $\vec{a} \perp \vec{b}$
(and of course $\vec{a} \neq \vec{b}$ as well)

Question 3. (17 points total)

Consider the space curve defined by the vector function

$$\mathbf{r}(t) = \langle 3\cos(2t),\ \ 2t,\ \ 3\sin(2t)\rangle \quad \text{with } 0 \le t \le 2\pi$$

(a) (6 points) Sketch the curve as neatly as possible (be sure to label your axes).



spiral along y-axis, radius is 3, travels up to $y = 4\pi$ along y-axis, spiraling around y-axis

(b) (3 points) Find an equation for a surface that this space curve lies on and describe the surface.

The curve spirals around the surface of a radius 3 cylinder aligned with the y-axis, with equation $x^2 + z^2 = 3^2$

(check that the curve lies on this surface,

as $\underbrace{(3\cos(2t))}_{x}^2 + \underbrace{(3\sin(2t))}_{z}^2 = 3^2\cos^2(2t) + 3^2\sin^2(2t)$

$= 3^2$ ✓

There are other possible surfaces, but this one is the simplest to describe

Question 3 continued

(Continue working with the space curve defined by $\mathbf{r}(t) = \langle 3\cos(2t),\ 2t,\ 3\sin(2t)\rangle$ with $0 \le t \le 2\pi$)

(c) (4 points) Find the length of this space curve.

We know $\text{arclength} = \int_a^b |\vec{r}\,'(t)|\,dt$

here $\vec{r}\,'(t) = \langle -6\sin(2t), 2, 6\cos(2t)\rangle$

and $|\vec{r}\,'(t)| = \sqrt{36\sin^2(2t) + 4 + 36\cos^2(2t)}$

$= \sqrt{36+4} = \sqrt{40}$

So along the space curve, with $0 \le t \le 2\pi$

$\text{arc-length} = \int_0^{2\pi} \sqrt{40}\,dt = 2\pi\sqrt{40}$ or $4\pi\sqrt{10}$

(d) (4 points) Give parametric equations for the tangent line to this space curve at the point $(3, 2\pi, 0)$.

The point $(3, 2\pi, 0)$ is $\vec{r}(\pi) = \langle 3\cos(2\pi), 2\pi, 3\sin(2\pi)\rangle$

i.e. $t = \pi$

here $\vec{r}\,'(\pi) = \langle -6\sin(2\pi), 2, 6\cos(2\pi)\rangle$.

$= \langle 0, 2, 6\rangle$

So point $= \langle 3, 2\pi, 0\rangle$, direction vector $= \langle 0, 2, 6\rangle$
(position vector)

line : $\langle 3, 2\pi, 0\rangle + t\langle 0, 2, 6\rangle = \langle 3, 2\pi + 2t, 6t\rangle$

or parametrically as $\begin{cases} x = 3 \\ y = 2\pi + 2t \\ z = 6t \end{cases}$

Question 4. (12 points total)

(a) (4 points) Consider the space curve that is defined as the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$. Explain why this space curve can be parametrized by the vector function:

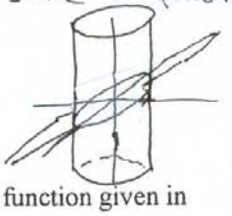
$$\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle \quad \text{with } 0 \le t \le 2\pi$$

Check to see that $\vec{r}(t)$ lies on both surfaces

for $x^2+y^2=1$ check $\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{\sin t} \rangle$

so $\cos^2 t + \sin^2 t = 1$ ✓

for $y=z$, sure the $y = \sin t \neq z = \sin t$ component



Now does it parametrize the whole intersection? yes, as $0 \le t \le 2\pi$ $\langle \cos t, \sin t \rangle$ makes a complete loop around the cylinder, so it covers the whole (elliptical) interssection

(b) (4 points) Now consider the vector function $\mathbf{p}(t) = \dfrac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$, where $\mathbf{r}(t)$ is the vector function given in part (a). Compute the components of vector function $\mathbf{p}(t)$.

$\vec{r}(t) = \sqrt{\cos^2 t + \sin^2 t + \sin^2 t} = \sqrt{1+\sin^2 t}$

so $\vec{p}(t) = \dfrac{\langle \cos t, \sin t, \sin t \rangle}{\sqrt{1+\sin^2 t}}$

$= \left\langle \dfrac{\cos t}{\sqrt{1+\sin^2 t}}, \dfrac{\sin t}{\sqrt{1+\sin^2 t}}, \dfrac{\sin t}{\sqrt{1+\sin^2 t}} \right\rangle$

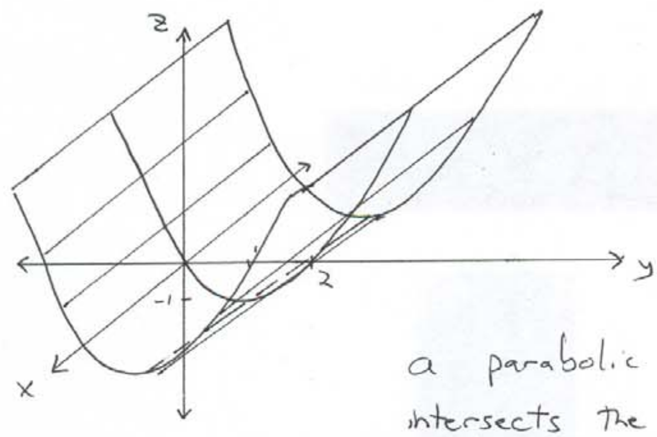
(c) (4 points) Now consider the space curve formed by the intersection of the same plane $y = z$ and the unit sphere (the sphere of radius one centered at the origin). Explain why this new space curve is parametrized by the vector function $\mathbf{p}(t)$.

Note $|\vec{p}(t)| = \left| \dfrac{\vec{r}(t)}{|\vec{r}(t)|} \right| = \dfrac{1}{|\vec{r}(t)|} |\vec{r}(t)| = 1$,

so $\vec{p}(t)$ lies on the unit sphere. Next, the y component still equals the z component of $\vec{p}(t)$, so $\vec{p}(t)$ lies on the intersection of the unit sphere and the $y=z$ plane. Once again, as $0 \le t \le 2\pi$ the $\vec{p}(t)$ space curve makes a complete loop around the intersection (which is a circle)

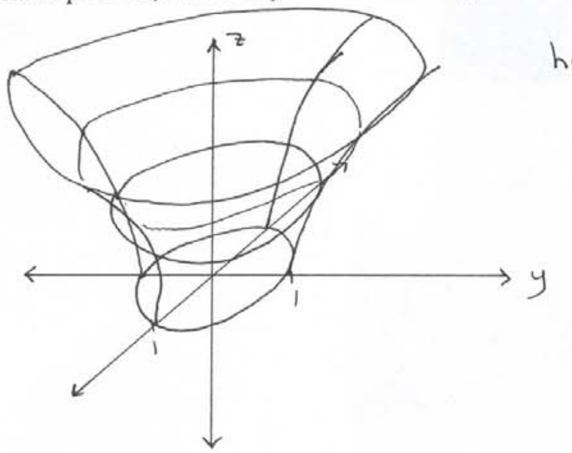
Question 5. (16 points total)

(a) (8 points) Sketch the graph of the function $f(x, y) = (y-1)^2 - 1$. Make your sketch as neat as possible, and be sure to label your axes carefully.



a parabolic cylinder
intersects the origin
(and the whole x-axis
as $f(x,0) = 0$ for all x)

(b) (8 points) Now sketch the graph of the function $f(x, y) = \sqrt{x^2 + y^2 - 1}$. Again, make your sketch as neat as possible, and label your axes carefully.



here $z^2 = x^2 + y^2 - 1$
(note $f(x,y) \geq 0$)
whenever z is constant
the cross-sections
are circles.
When $x = 0$
$z^2 = y^2 - 1$,
or $y^2 - z^2 = 1$,
a hyperbola
in the y-z plane
(with $z > 0$)

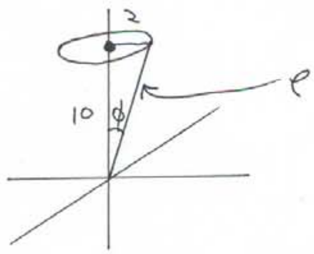
(also get $x^2 - z^2 = 1$
in xz plane
cross-section)

(net result is
half a hyperboloid
of one sheet)

Question 6. (14 points total)

A moth is flying around a lightbulb hanging from the ceiling of a room. The lightbulb is 10 feet above the floor and the moth stays exactly 10 feet above the floor as it flies around the lightbulb. The moth flies in a circular path, at constant speed, always exactly 2 feet away from the center of the lightbulb.

(a) (4 points) Write down the set of equations that describe the circle traced out by the moth in spherical coordinates. Consider the origin to be the point on the floor directly below the lightbulb.



We need ρ, θ and ϕ.
Now θ can take on any value,
but ρ is always $\sqrt{10^2+2^2} = \sqrt{104}$
and $\tan\phi = \frac{2}{10}$, so $\phi = \tan^{-1}\left(\frac{1}{5}\right)$

So Equations : $\begin{cases} \rho = \sqrt{104} \\ \phi = \tan^{-1}\left(\frac{1}{5}\right) \end{cases}$

(b) (4 points) Write down the equations giving the moth's circular path in cylindrical coordinates, and then give a parametrization of the moth's path in cylindrical coordinates.

In cylindrical coordinates, even easier.
Here we need r, θ and z. Again θ can take on any value, but $z = 10$ for the whole circle, and r is always 2 (distance from z-axis)
so set of equations : $\begin{cases} r = 2 \\ z = 10 \end{cases}$

To parametrize the curve we need to make θ take on all the values from 0 to 2π, e.g. $\begin{cases} r = 2 \\ z = 10 \\ \theta = t \end{cases}$ with $-\infty < t < \infty$

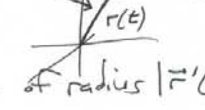
(c) (6 points) Suppose that the moth's flight path is given by a vector function **r**(t), with parameter t representing time. Given what you know about the moth's flight answer the following questions:

(i) Is it possible to determine **r'**(t) • **k**? If so, what is it?

Sure, $\vec{r}(t) = \langle *_1, *_2, 10 \rangle$ (some functions in for "$*$")
so $\vec{r}\,'(t) = \langle @_1, @_2, 0 \rangle$ $@$ = derivatives of $*$ functions
The point is $\vec{r}\,'(t) \cdot \langle 0,0,1 \rangle = 0$. Or note $\vec{r}\,'(t)$ is tangent to the circular path, and so is parallel to x-y plane, so is $\perp$ to z-axis, (so $\perp \vec{k}$) so dot product $= 0$

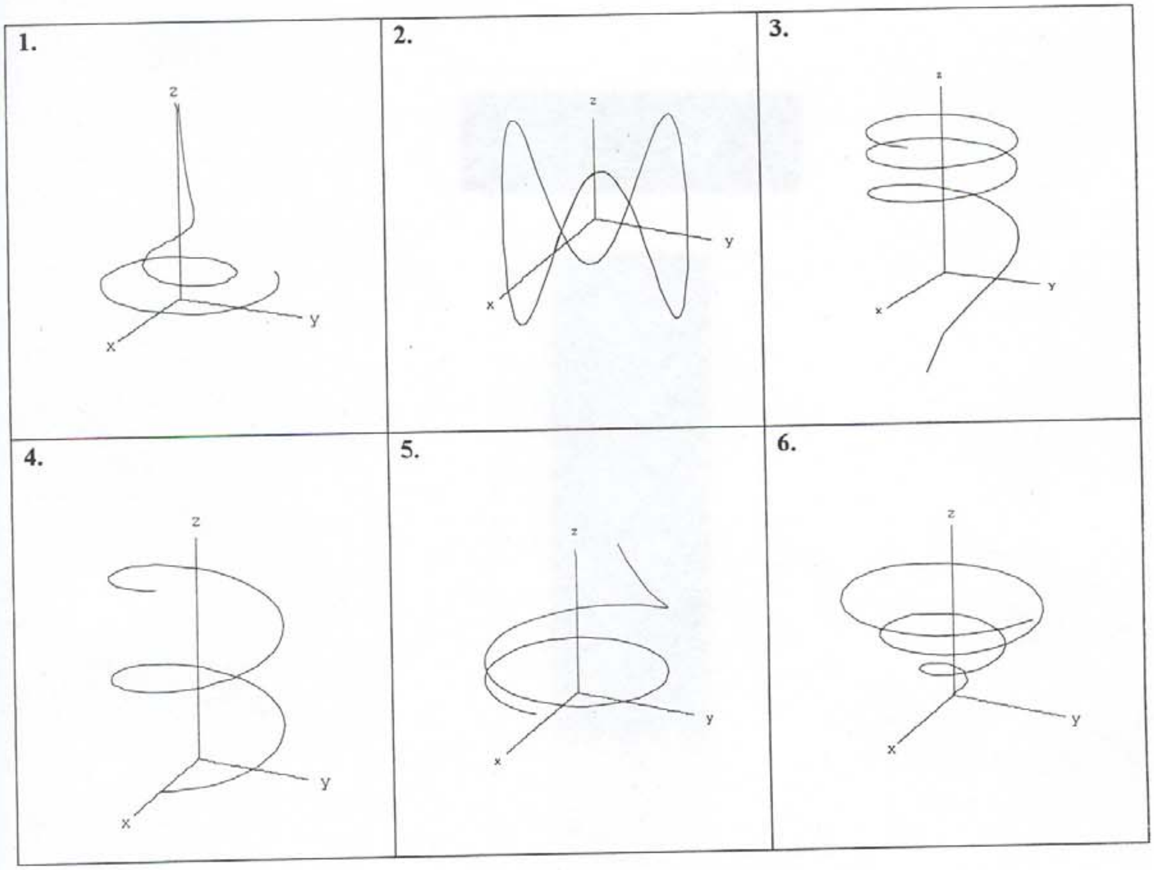
(ii) Describe the curve parametrized by **r'**(t) x **k**

Here note $|\vec{r}\,'(t) \times \vec{k}| = |\vec{r}\,'(t)|\,|k| \sin\theta$
$= |\vec{r}\,'(t)| = \text{constant}$
as moth flies at constant speed.
Now note $\vec{r}\,'(t)$ is parallel to the xy plane, so $\vec{r}\,'(t) \times \vec{k}$ is $\perp$ to $\vec{k}$, in xy plane as well. As $\vec{r}(t)$ traces out a circular path at height 10, then, $\vec{r}\,'(t) \times \vec{k}$ traces out a circular path in the xy plane of radius $|\vec{r}\,'(t)|$



Question 7. (12 points total, 2 points each)

Round and round they go… Match the following space curves to the appropriate vector functions (and yes, we've been polite and made everything match up). Write the number of the appropriate space curve next to the correct vector function.



3 $\mathbf{r}(t) = \langle \cos t, \sin t, \ln t \rangle$	1 $\mathbf{r}(t) = \left\langle t\cos t, t\sin t, \frac{1}{t} \right\rangle$	2 $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(3t) \rangle$
6 $\mathbf{r}(t) = \langle t\cos t, t\sin t, t \rangle$	4 $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$	5 $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{1}{t} \right\rangle$