Nash's problem given with the encouraging words

"It might solve it in a few months. Most of you however will need a life-time".

to a multivariable calculus class in the movie "A beautiful mind" is:

NASH's PROBLEM. Find a subset X of $\mathbf{R}^3$ with the property that V/W is 8-dimensional, where V is the set of vector fields F on $\mathbf{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and where W is the set of vector fields F which are conservative $F = \nabla f$.

The SOLUTION OF NASH's PROBLEM:
Let X be the union of 8 distinct parallel lines $X = \bigcup_{i=1}^{8} \{(x_i, y_i, z) \mid -\infty < z < \infty\}$ in $\mathbf{R}^3$.
The vector fields $F_i(x,y,z) = (-(y-y_i)/((x-x_i)^2+(y-y_i)^2), (x-x_i)/((x-x_i)^2+(y-y_i)^2), 0)$.
satisfy $\text{curl}(F_i) = 0$ outside X.
Every vector field which satisfies $\text{curl}(F) = 0$ in $\mathbf{R}^3 \setminus X$ can be written as

$$F = G + \sum_{i=1}^{8} a_i F_i \ ,$$

where a_i are some real numbers and where $G = \nabla g$ is a vector field which is a gradient.

BACKGROUND. Even so the problem can be posed and solved in a multi-variable course (as in this summer school), the problem rather belongs to a more advanced **algebraic topology course**, to be fully appreciated. Nash's problem is the **inverse cohomology problem** to find a manifold M with a 8-dimensional fundamental group.

DE RHAM'S THEOREM. A special case of "de Rham theorem" states that on a manifold M, the vector space of all 1-forms F satisfying $dF = 0$ modulo the space of all 1-forms F which are of the form $F = dG$ is the same as the first cohomology group $H^1(M)$, which is equal to the fundamental group of M.

The dimension of the fundamental group of M is the maximal number of closed paths, which you can find in M so that no path can be deformed inside M to any other other path in that family.

In 3 dimensons, 1-forms can be associated with vector fields. For every 1-form, dF is a 2-form which is the curl of F. In 3 dimensions, 2-forms can be identified with vector fields. If G is a 0-form, a smooth function on M, then dG is the gradient of G.

To find a space X in Nash's problem, one has to find a manifold with a 8-dimensional fundamental group. Taking away 8 lines from three dimensional space is one of the possibilities. A closed path which winds around one of the lines and no other line can not be deformed to a point (otherwise, people could steel your bike chained to a pole), nor can it be deformed to a path which winds around an other line.

The solution with lines is not unique. One could take for example the union of 8 closed arbitrary unlinked and unknotted curves in space. It would be hard however to come up in general with explicit vectorfields F_i, whose existence is assured by de Rham's theorem.