**11/2/2016 Second Hourly Practice IV** **Math 21a, Fall 2016**

| Name: |
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| TTH 10 George Melvin |
| TTH 11:30 Jake Marcinek |
| TTH 11:30 George Melvin |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

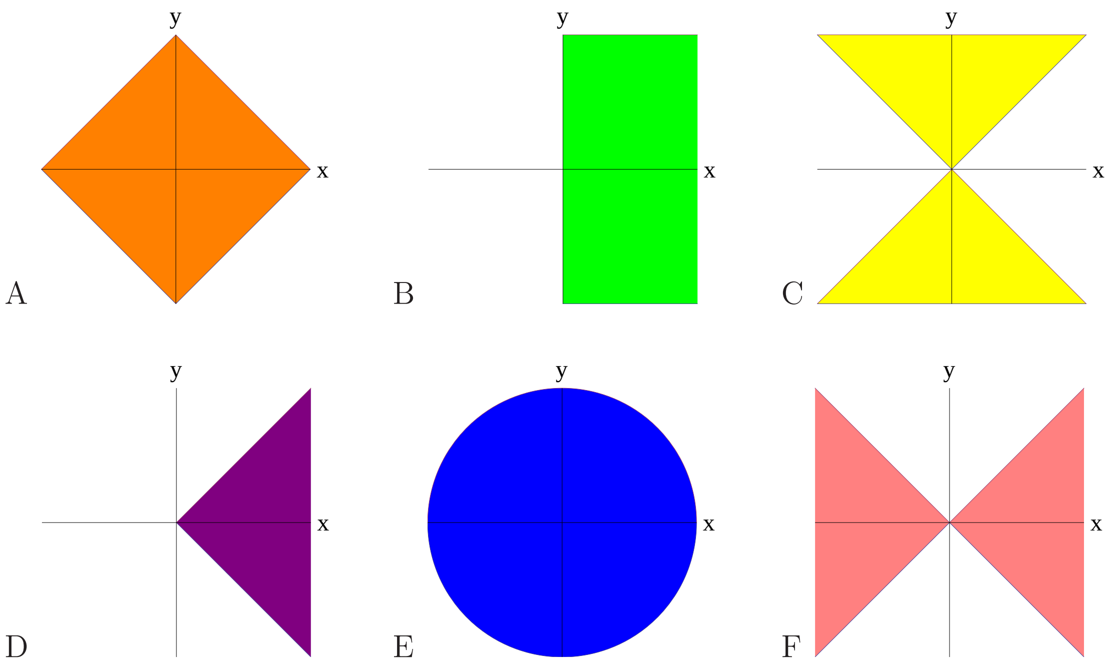
| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) True/False questions (20 points), no justifications needed

| | | | |
|---|---|---|---|
| 1) | T | F | If $\vec{r}(t)$ is a space curve satisfying $\vec{r}\,'(0) = 0$ and $f(x, y, z)$ is a function of three variables then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at $t = 0$. |
| 2) | T | F | The integral $\int\int_R 1\, dxdy$ is the area of the region $R$ in the $xy$-plane. |
| 3) | T | F | If $f(x, y)$ is a linear function in $x, y$, then $D_{\vec{u}} f(x, y)$ is independent of $\vec{u}$. |
| 4) | T | F | If $f(x, y)$ is a linear function in $x, y$, then $D_{\vec{u}} f(x, y)$ is independent of $(x, y)$. |
| 5) | T | F | If $(0, 0)$ is a saddle point of $f(x, y)$ it is possible that $(0, 0)$ is a minimum of $f(x, y)$ under the constraint $x = y$. |
| 6) | T | F | The equation $f_{xy}(x, y) = 0$ is an example of a partial differential equation. |
| 7) | T | F | The linearization of $f(x, y) = 4 + x^3 + y^3$ at $(x_0, y_0) = (0, 0)$ is $L(x, y) = 4 + 3x^2 + 3y^2$. |
| 8) | T | F | Assume $(1, 1)$ is a saddle point of $f(x, y)$. Then $D_{\vec{v}} f(1, 1)$ takes both positive and negative values as $\vec{v}$ varies over all directions. |
| 9) | T | F | The integral $\int_{\pi/2}^{\pi} \int_0^2 r\, drd\theta$ is equal to $\pi$. |
| 10) | T | F | If $\|\nabla f(0, 0)\| = 1$, then there is a direction in which the slope of the graph of $f$ at $(0, 0)$ is 1. |
| 11) | T | F | The vector $\nabla f(a, b)$ is a vector in space orthogonal to surface defined by $z = f(x, y)$ at the point $(a, b)$. |
| 12) | T | F | If $f(x, y, z) = 1$ defines $y$ as a function of $x$ and $z$, then $\partial y(x, z)/\partial x = -f_x(x, y, z)/f_y(x, y, z)$. |
| 13) | T | F | In a constrained optimization problem it is possible that the Lagrange multiplier $\lambda$ is 0. |
| 14) | T | F | The area $\int\int_R \|\vec{r}_u \times \vec{r}_v\|\, dudv$ of a surface is independent of the parametrization. |
| 15) | T | F | The function $f(x, y) = x^6 + y^6 - x^5$ has a global minimum in the plane. |
| 16) | T | F | The area of a graph $z = f(x, y)$ where $(x, y)$ is in a region $R$ is the integral $\int\int_R \|f_x \times f_y\|\, dxdy$. |
| 17) | T | F | The gradient of a function $f(x, y)$ of two variables can be written as $\langle D_{\vec{i}} f(x, y), D_{\vec{j}} f(x, y)\rangle$, where $\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$. |
| 18) | T | F | The length of the gradient of $f$ at a critical point is positive if the discriminant $D(x, y) = f_{xx} f_{yy} - f_{xy}^2$ is strictly positive. |
| 19) | T | F | If $f(0, 0) = 0$ and $f(1, 0) = 2$ then there is a point on the line segment between $(0, 0)$ and $(1, 0)$, where the gradient has length at least 2. |
| 20) | T | F | The tangent plane of the surface $-x^2 - y^2 + z^2 = 1$ at $(0, 0, 1)$ intersects the surface at exactly one point. |

Problem 2) (10 points)

a) (6 points) Match the regions with the integrals. Each integral matches exactly one region $A - F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-\pi}^{\pi} \int_{-\lvert y\rvert}^{\lvert y\rvert} f(x,y)\ dxdy$ |
| | $\int_{-\pi}^{\pi} \int_{0}^{\pi} f(r,\theta) r\ drd\theta$ |
| | $\int_{-\pi}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\ dydx$ |
| | $\int_{0}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\ dydx$ |
| | $\int_{-\pi}^{\pi} \int_{0}^{\pi} f(x,y)\ dxdy$ |
| | $\int_{-\pi}^{\pi} \int_{-\pi+\lvert x\rvert}^{\pi-\lvert x\rvert} f(x,y)\ dydx$ |

b) (4 points) Name the partial differential equations correctly. Each equation appears once to the left.

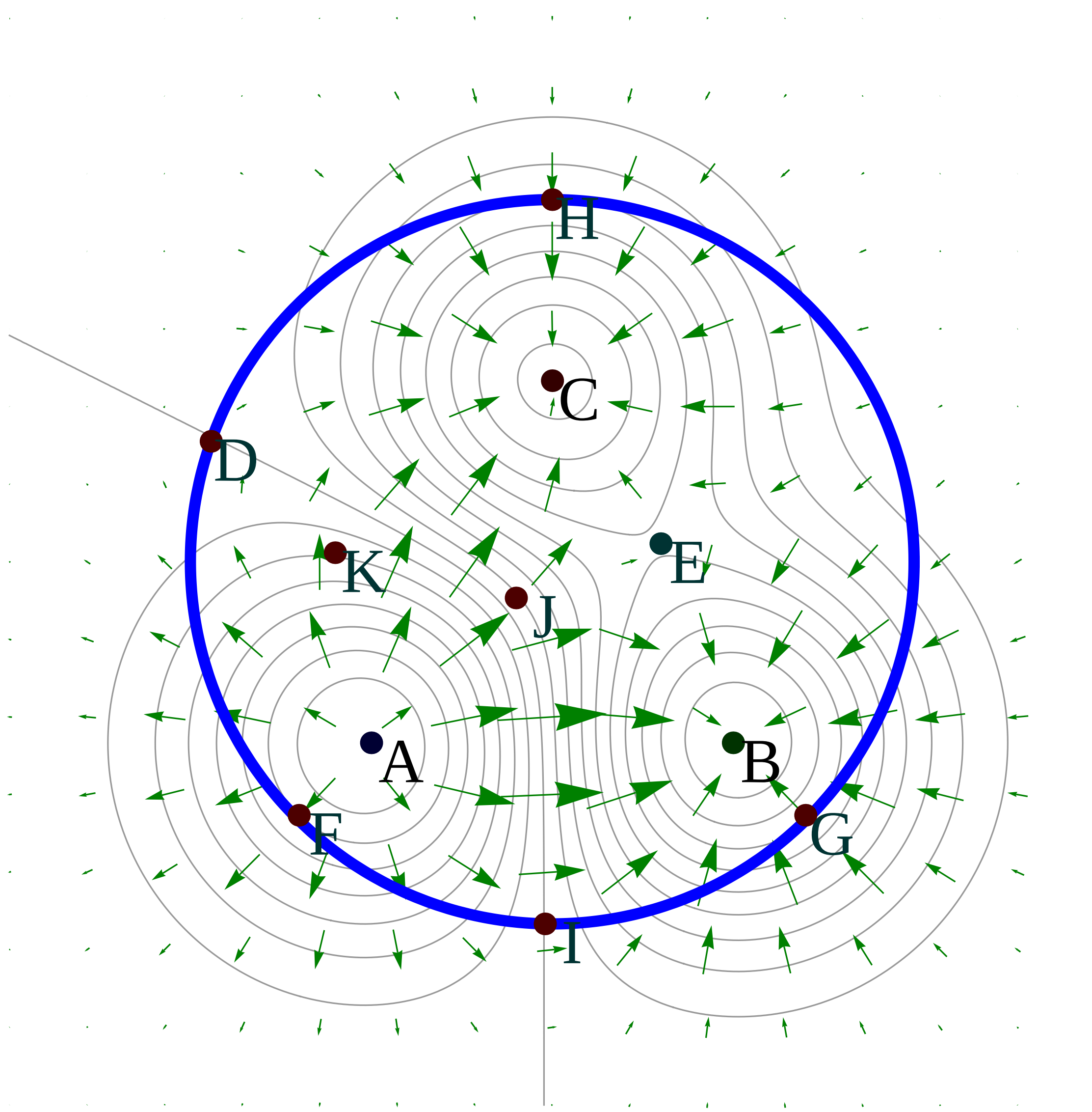
| Fill in 1-4 | Order |
|---|---|
| | Burgers |
| | Transport |
| | Heat |
| | Wave |

| Equation Number | PDE |
|---|---|
| 1 | $u_x - u_y = 0$ |
| 2 | $u_{xx} - u_{yy} = 0$ |
| 3 | $u_x - u_{yy} = 0$ |
| 4 | $u_x + uu_x - u_{xx} = 0$ |

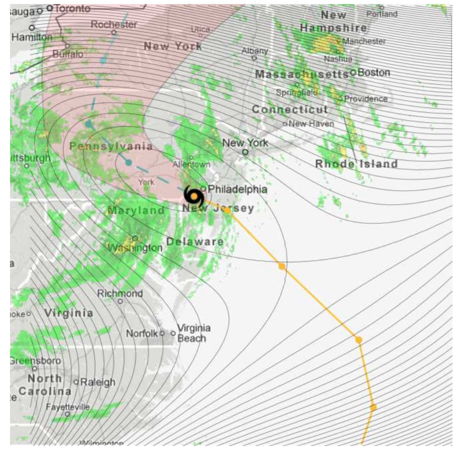
| Problem 3) (10 points) |
|---|

(10 points) Let's label some points in the following contour map of a function $f(x,y)$ indicating the height of a region. The arrows indicate the gradient $\nabla f(x,y)$ at the point. Each of the 11 selected points appears each exactly once.

| Enter A-K | description |
|---|---|
| | a local minimum of $f(x,y)$ inside the circle |
| | a saddle point of $f(x,y)$ inside the circle |
| | a point, where $f_x \neq 0$ and $f_y = 0$ |
| | a point, where $f_x = 0$ and $f_y > 0$ |
| | a point, where $f_x = 0$ and $f_y < 0$ |
| | a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$. |
| | the lowest point on the circle |
| | the highest point on the circle |
| | the local but not global maximum inside or on the circle |
| | the global maximum inside or on the circle |
| | the steepest point inside the circle |



Problem 4) (10 points)

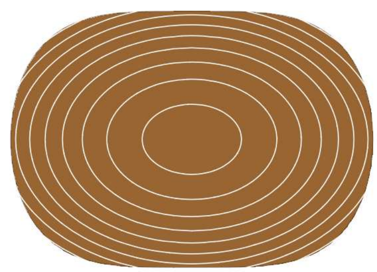



On October 30, 2012, the wind speed of Hurricane Sandy was given by the function

$$f(x,y) = 60 - x^3 + 3xy + y^3 .$$

Classify the critical points (maxima, minima and saddle points) of this function. Compute also the values of $f$ at these points.

Problem 5) (10 points)



Use the second derivative test and the method of Lagrange multipliers to find the global maximum and minimum of the sugar concentration $f(x,y) = 10 + x^2 + 2y^2$ on a cake given by

$$g(x,y) = x^4 + 4y^2 \leq 4 .$$

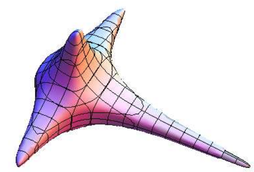
Note that this means you have to look both inside the cake and on the boundary.

Problem 6) (10 points)

a) (5 points) A seed of "Tribulus terrestris" has the shape

$$x^2 + y^2 + z^2 + x^4y^4 + x^4z^4 + y^4z^4 - 9z = 21$$

Find the tangent plane at $(1, 1, 2)$.



b) (5 points) The seed intersects with the $xy$-plane in a curve

$$x^2 + y^2 + x^4y^4 = 21 \ .$$

Find the tangent line to this curve at $(1, 2)$.

Problem 7) (10 points)

Picture by Ellen van Deelen,

South west News service (UK)

Let $f(x, y)$ model the time that it takes a rat to complete a maze of length $x$ given that the rat has already run the maze $y$ times. We know $f_y(10, 20) = -5$ and $f_x(10, 20) = 1$ as well as $f(10, 20) = 45$. Use this to estimate $f(11, 18)$.

Problem 8) (10 points)

a) (5 points) Find the double integral

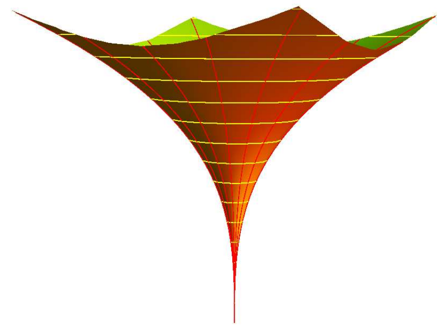
$$\int\int_R x \ dydx \ ,$$

where $R$ is the region obtained by intersecting $x \leq |y|$ with $x^2 + y^2 \leq 1$.

b) (5 points) The square $\sin^2(x)/x^2$ of the sync function $\sin(x)/x$ does not have a known antiderivative. Compute nevertheless the integral

$$\int_0^{\pi^2/4} \int_{\sqrt{y}}^{\pi/2} \frac{\sin^2(x)}{x^2} \, dx \, dy \ .$$

Problem 9) (10 points)



Find the surface area of the surface of revolution $x^2 + y^2 = z^6$ where $0 \leq z \leq 1$. The surface is parametrized by

$$\vec{r}(t,z) = \langle z^3 \cos(t), z^3 \sin(t), z \rangle$$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$.

Problem 10) (10 points)


It turns out that there is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{x^7}{\sqrt{x} - x^2} \, dx \, dy \, .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{10} \, dxdy \, .$$