

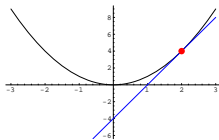
## TANGENT PLANES

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REMINDER: TANGENT LINE. Because  $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$  is perpendicular to the level curve  $f(x, y) = c$  through  $(x_0, y_0)$ , the equation for the tangent line is

$$ax + by = d, \quad a = f_x(x_0, y_0), \quad b = f_y(x_0, y_0), \quad d = ax_0 + by_0$$

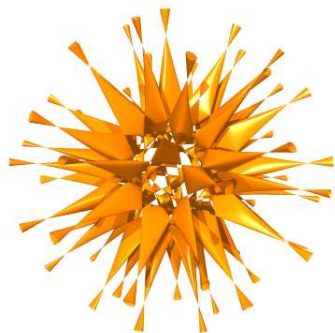
Example: Find the tangent to the graph of the function  $g(x) = x^2$  at the point  $(2, 4)$ . Solution: the level curve  $f(x, y) = y - x^2 = 0$  is the graph of a function  $g(x) = x^2$  and the tangent at a point  $(2, g(2)) = (2, 4)$  is obtained by computing the gradient  $\langle a, b \rangle = \nabla f(2, 4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$  and forming  $-4x + y = d$ , where  $d = -4 \cdot 2 + 1 \cdot 4 = -4$ . The answer is  $-4x + y = -4$  which is the line  $y = 4x - 4$  of slope 4. Graphs of 1D functions are curves in the plane, you have computed tangents in single variable calculus.



GRADIENT IN 3D. If  $f(x, y, z)$  is a function of three variables, then  $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$  is called the **gradient** of  $f$ .

POTENTIAL AND FORCE. Force fields  $F$  in nature often are gradients of a function  $U(x, y, z)$ . The function  $U$  is called a **potential** of  $F$  or the potential energy.

EXAMPLE. If  $U(x, y, z) = 1/|x|$ , then  $\nabla U(x, y, z) = -x/|x|^3$ . The function  $U(x, y, z)$  is the **Coulomb potential** and  $\nabla U$  is the **Coulomb force**. The gravitational force has the same structure but a different constant. While much weaker, it is more effective because it only appears as an attractive force, while electric forces can be both attractive and repelling.

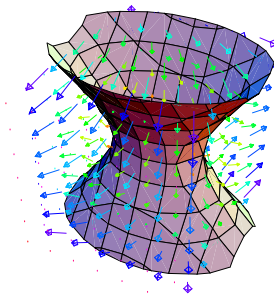


LEVEL SURFACES. If  $f(x, y, z)$  is a function of three variables, then  $f(x, y, z) = C$  is a surface called a level surface of  $f$ . The picture to the right shows the Barth surface  $(3 + 5t)(-1 + x^2 + y^2 + z^2)^2 (-2 + t + x^2 + y^2 + z^2)^2 8(x^2 - t^4 y^2)(-t^4 x^2 + z^2) (y^2 - t^4 z^2) (x^4 - 2x^2 y^2 + y^4 - 2x^2 z^2 - 2y^2 z^2 + z^4) = 0$ , where  $t = (\sqrt{5} - 1)/2$  is the golden ratio.

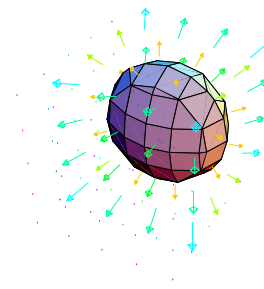
ORTHOGONALITY OF GRADIENT. We have seen that the gradient  $\nabla f(x, y)$  is perpendicular to the level curve  $f(x, y) = c$ . This is also true in 3 dimensions:

The gradient  $\nabla f(x, y, z)$  is normal to the level surface  $f(x, y, z)$ .

The argument is the same as in 2 dimensions: take a curve  $\vec{r}(t)$  on the level surface. Then  $\frac{d}{dt} f(\vec{r}(t)) = 0$ . The chain rule tells from this that  $\nabla f(x, y, z)$  is perpendicular to the velocity vector  $\vec{r}'(t)$ . Having  $\nabla f$  perpendicular to all tangent velocity vectors on the surfaces means that it is orthogonal to the surface.

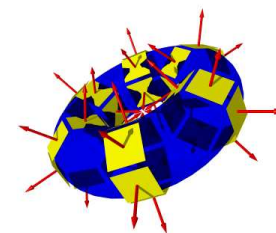


EXAMPLE. The gradient of  $f(x, y, z) = x^2 + 2y^2 + z^2$  at a point  $(x, y, z)$  is  $(2x, 4y, 2z)$ . It illustrates well that going into the direction of the gradient **increases** the value of the function.



TANGENT PLANE. Because  $\vec{n} = \nabla f(x_0, y_0, z_0) = \langle a, b, c \rangle$  is perpendicular to the level surface  $f(x, y, z) = C$  through  $(x_0, y_0, z_0)$ , the equation for the tangent plane is

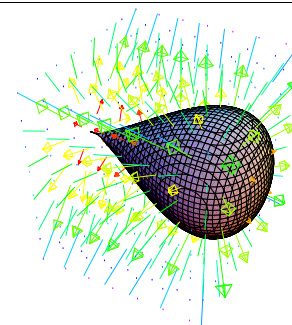
$$ax + by + cz = d, \quad \langle a, b, c \rangle = \nabla f(x_0, y_0, z_0), \quad d = ax_0 + by_0 + cz_0.$$



EXAMPLE. Find the general formula for the tangent plane at a point  $(x, y, z)$  of the Barth surface. Just kidding ... Note however that computing this would be no big deal with the help of a computer algebra system like Mathematica. Lets look instead at the quartic surface

$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

which is also called the "piriform" or "pair shaped surface". What is the equation for the tangent plane at the point  $P = (2, 2, 2)$ ? We get  $\langle a, b, c \rangle = \langle 20, 4, 4 \rangle$  and so the equation of the plane  $20x + 4y + 4z = 56$ .



EXAMPLE. An important example of a level surface is  $g(x, y, z) = z - f(x, y)$  which is the graph of a function of two variables. The gradient of  $g$  is  $\nabla g = \langle -f_x, -f_y, 1 \rangle$ . This allows us to find the equation of the tangent plane at a point.

Homework: What is the relation between the gradient of  $f$  in the plane and the gradient of  $g$  in space?

