CURVATURE — Math21a

HOMEWORK: Section 12.4: 10, 46, (and on a different topic 13.1: 38, 46, 64)

CURVATURE. If $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ **unit tangent vector**, define the **curvature** at the point $\vec{r}(t)$ as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} .$$

CURVATURE FORMULA. The following formula for the curvature sheds more light on the curvature and is often easier to compute:

$$\boxed{\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

The same formula holds for curves in the plane if we define the cross product in the plane as $(a, b)\times(c, d) = ad-bc$.

EXAMPLE. CIRCLE

$\vec{r}(t) = (r\cos(t), r\sin(t))$.
$\vec{r}\,'(t) = (-r\sin(t), r\cos(t))$.
$|\vec{r}\,'(t)| = r$.
$\vec{T}(t) = (-\sin(t), \cos(t))$.
$\vec{r}\,''(t) = (-r\cos(t), -r\sin(t))$.
$\vec{T}'(t) = (-\cos(t), -\sin(t))$.
$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = 1/r$.

EXAMPLE. HELIX

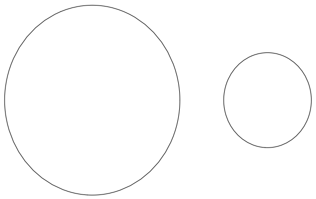
$\vec{r}(t) = (\cos(t), \sin(t), t)$.
$\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$.
$|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.
$\vec{T}(t) = (-\sin(t), \cos(t), 1)/\sqrt{2}$.
$\vec{r}\,''(t) = (-\cos(t), -\sin(t), 0)$.
$\vec{T}'(t) = (-\cos(t), \sin(t), 0)/\sqrt{2}$.
$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = 1/2$.

INTERPRETATION.

If $s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$, then $s'(t) = ds/dt = |\vec{r}\,'(t)|$. Because $\vec{T}'(t) = d\vec{T}/dt = d\vec{T}/ds \cdot ds/dt$, we have $|d\vec{T}/ds| = |\vec{T}'(t)|/|\vec{r}\,'(t)| = \kappa(t)$.

> "The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1."

A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.



Small curvature $\kappa = 1/r = 1/2$

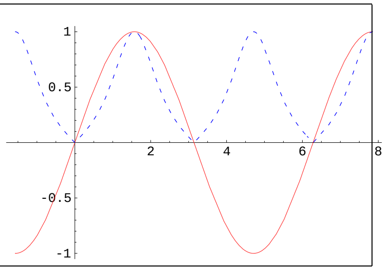
Large curvature $\kappa = 1/r = 2$

CURVATURE OF A GRAPH.

The curve $\vec{r}(t) = (t, f(t))$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$ and after some simplification

$$\boxed{\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3}$$

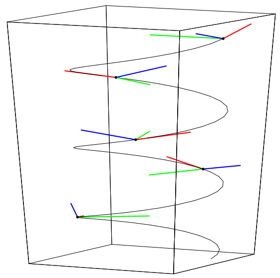
EXAMPLE. $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.



TANGENT/NORMAL/BINORMAL.

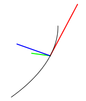
$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ **tangent vector**
$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ **unit normal vector**
$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ **binormal vector**

Because $\vec{T}(t)\cdot\vec{T}(t) = 1$, we get after differentiation $\vec{T}'(t)\cdot\vec{T}(t) = 0$ and $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$.



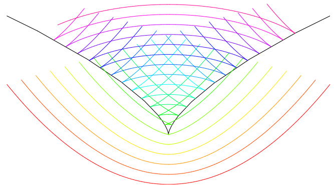
> The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

Note. In order that $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ exist, we need that $\vec{r}\,'(t)$ is not zero.



WHERE IS CURVATURE NEEDED?

OPTICS. If a curve $\vec{r}(t)$ represents a wavefront and $\vec{n}(t)$ is a unit vector normal to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.



HISTORY.

Aristotle: (350 BC) distinguishes between straight lines, circles and "mixed behavior"
Oresme: (14'th century): measure of twist called "curvitas"
Kepler: (15'th century): circle of curvature.
Huygens: (16'th century): evolutes and involutes in connection with optics.
Newton: (17'th century): circle has constant curvature inversely proportional to radius. (using infinitesimals)
Simpson: (17'th century): string construction of evolutes, description using fluxions.
Euler: (17'th century): first formulas of curvature using second derivatives.
Gauss: (18'th century): modern description, higher dimensional versions.

COMPUTING CURVATURE WITH MATHEMATICA

```
x[t_] := Cos[3t];
y[t_] := Sin[2t];
r[t_] := {x[t], y[t]};
dr[t_] := D[r[s], s]/.s− > t;
L[{a_,b_}] := Sqrt[a^2 + b^2];
T[t_] := dr[t]/L[dr[t]];
dT[t_] := D[T[s], s]/.s− > t;
kappa[t_] := L[dT[t]]/L[dr[t]]
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