Last Name: Answer Key

First Name: ______________________

Math 21a Exam #2: Wednesday, April 11, 2001

SECTION (CIRCLE ONE):

Robert Winters Michael Simonetti (CA) MWF 10-11	Robert Winters Lisa Carlivati (CA) MWF 11-12	Paul Bamberg Geoff Werner-Allen (CA) MWF 11-12 (Physics)
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Question	Points	Score
1	16	
2	14	
3	14	
4	14	
5	14	
6	14	
7	14	
Total	100	

The time allotted for this exam is 90 minutes.

Justify your answers carefully. No partial credit can be given for unsubstantiated answers.

If more space is needed, use the back of the previous page and make note of this.

Please write neatly. Answers which are deemed illegible by the grader will not receive credit.

No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.

(1) The President has opened up a wildlife preserve in Alaska for oil drilling. In retaliation, Democrats introduce a bill to allow oil drilling on the Ellipse, a park between the White House and the Washington Monument bounded by the curve $x^2 + 4y^2 = 100$. Each political party is to be given one drilling site.

The President consults geologists from Texas, who inform him that the value V of the oil from a well drilled on the Ellipse will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) He thereupon signs the bill into law and instructs you, his Secretary of Energy, to find the coordinates x and y for the most valuable drilling site(s) (for the Republicans) and of the least valuable site(s) (for the Democrats) and to tell him the maximum and minimum values. Do so.

To find local max, mins inspect critical points:

$\nabla V = \langle -2x, 18-2y \rangle$ $\nabla V = \langle 0,0 \rangle$ at $(0,9)$ ↳ outside Ellipse.

so no local max/mins inside Ellipse – check boundary,

$x^2+4y^2=100$. parametrize this ellipse as

$\vec{r}(t) = \langle 10\cos t, 5\sin t \rangle$, $0 \le t \le 2\pi$ (check that $(10\cos t)^2 + 4(5\sin t)^2 = 100$ ✓ $\to x^2 + 4y^2$)

so on boundary $V = 200 + 18y - x^2 - y^2$

$= 200 + 18(5\sin t) - (10\cos t)^2 - (5\sin t)^2$

$= 200 + 90\sin t - 100\cos^2 t - 25\sin^2 t$

find max/mins directly using usual single variable approach

$\frac{dV}{dt} = 90\cos t - 200\cos t(-\sin t) - 50\sin t \cdot \cos t$

$= 90\cos t + 150\cos t\sin t$ set $=0$, either $\cos t = 0 \Rightarrow t = \pi/2$ or $3\pi/2$

or $90 + 150\sin t = 0 \Rightarrow \sin t = -\frac{90}{150} = -\frac{3}{5}$

actually just need coordinates $\langle 10\cos t, 5\sin t \rangle$ so look at points with $\cos t = 0$, $\sin t = 1 \Rightarrow$ points $(0, \pm 5)$ or $\sin t = -3/5 \Rightarrow \cos t = \sqrt{1-\sin^2 t} = \pm 4/5$

i.e. points $(\pm 10 \cdot 4/5, -5 \cdot 3/5) = (\pm 8, -3)$. Now check V on each point.

$V(0,5) = 265$ $V(0,-5) = 85$, $V(\pm 8,-3) = 73$

→ so most valuable site is on boundary at $(0,5)$, least valuable at either $(8,-3)$ or $(-8,-3)$.

(b) The President also wants to know whether the Republicans could do even better if allowed to drill outside the Ellipse? Determine if there is a drilling site with even greater V, and if so, where it is located.

(b) Now check out the point where $\nabla V = \vec{0}$: $(0,9)$ Since V is a downward pointing paraboloid function we suspect $V(0,9) = 281$ is an absolute max value. In any case, it's more valuable than the site found in (a).

(2) The function $F(x,y) = x^2y - 4xy + 3x^2 + \frac{1}{2}y^2$ has three stationary points, at $x = 0$, 1, and 5.

(a) Find the values of y at these three stationary points.

(b) Classify each stationary point as a maximum, minimum, or saddle point.

Answer (a) we know a stationary point (or critical point) occurs where $\nabla F = \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle = \langle 0,0 \rangle$, so ANT we simply solve:

$$\nabla F = \langle 2xy - 4y + 6x,\ x^2 - 4x + y \rangle$$

when $x=0$ $\nabla F(0,y) = \langle -4y, 2y \rangle$ so $y=0$ to make $\nabla F(0,0) = \langle 0,0 \rangle$

when $x=1$ $\nabla F(1,y) = \langle -2y+6, -3+y \rangle$ so $y=3$

when $x=5$ $\nabla F(5,y) = \langle 6y+30, 5+y \rangle$ so $y=-5$

(b) 2nd derivatives test

"D" $= F_{xx} \cdot F_{yy} - (F_{xy})^2 = (2y+6)(1) - (2x-4)^2$

at $(0,0)$ $D = 6 - 16 < 0$ $\Rightarrow$ saddle point

at $(1,3)$ $D = 12 - 4 > 0$ and $F_{xx}(1,3) = 12 > 0$ $\Rightarrow$ local minimum

at $(5,-5)$ $D = -4 - 36 < 0$ $\Rightarrow$ saddle point

(3) A function $F(x, y)$ is given by the formula $F(x, y) = g(x^2 + y^2)$, where g is a twice-differentiable function of one variable. Express $\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2}$ in terms of x, y, and the first and second derivatives of g.

$$\frac{\partial F}{\partial x} = g'(x^2+y^2)\cdot(2x) \quad \text{(chain rule)}$$

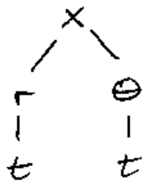
$$\frac{\partial^2 F}{\partial x^2} = g'(x^2+y^2)\cdot 2 + (2x)\cdot g''(x^2+y^2)\cdot(2x) \quad \text{chain rule w/ product rule}$$

likewise $$\frac{\partial F}{\partial y} = g'(x^2+y^2)(2y)$$

$$\frac{\partial^2 F}{\partial y^2} = g'(x^2+y^2)\cdot(2) + (2y)\, g''(x^2+y^2)\cdot(2y)$$

so $$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4g'(x^2+y^2) + (4x^2+4y^2)\, g''(x^2+y^2)$$

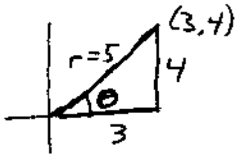
(4) Cartesian and polar coordinates are related by $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix}$. Suppose that at a certain instant in time, a particle is located at $(x, y) = (3, 4)$, and its polar coordinates are changing as specified by $\frac{dr}{dt} = 2$ and $\frac{d\theta}{dt} = 1$. Use the chain rule to calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for the particle at this instant.



so $\frac{dx}{dt} = \frac{\partial x}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial x}{\partial \theta} \frac{d\theta}{dt}$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$

when $(x, y) = (3, 4) \qquad r = \sqrt{3^2 + 4^2} = 5$



we don't need θ precisely, we just need to know cos θ and sin θ, which come from the diagram:

$\cos\theta = \frac{3}{5} \qquad \sin\theta = \frac{4}{5}$

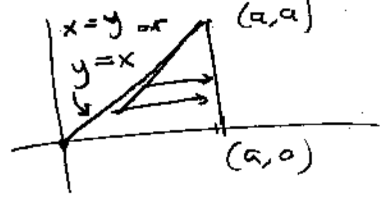
so $\frac{dx}{dt} = \frac{3}{5} \cdot 2 + \left(-5 \cdot \frac{4}{5}\right) \cdot 1 = -\frac{14}{5}$

$\frac{dy}{dt} = \frac{\partial y}{\partial r} \frac{dr}{dt} + \frac{\partial y}{\partial \theta} \cdot \frac{d\theta}{dt} \qquad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

so $\frac{dy}{dt} = \frac{4}{5} \cdot 2 + \left(5 \cdot \frac{3}{5}\right) \cdot 1 = \frac{23}{5}$

(5) (a) Using Cartesian coordinates, evaluate the integral of the function $x^2 + y^2$ over the right triangle with vertices $(x, y) = (0, 0)$, $(a, 0)$, and (a, a).

Triangle :



integrating wrt x first $y \le x \le a$ while $0 \le y \le a$

so get $\int_0^a \int_y^a (x^2+y^2)\,dx\,dy$

$$= \int_0^a \left(\frac{x^3}{3} + y^2 x \Big|_{x=y}^{x=a} \right) dy$$

$$= \int_0^a \left(\frac{a^3}{3} + a y^2 - \frac{y^3}{3} - y^3 \right) dy$$

$$= \left(\frac{a^3}{3} y + \frac{a}{3} y^3 - \frac{y^4}{12} - \frac{y^4}{4} \right) \Big|_0^a$$

$$= a^4 \left(\frac{1}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{4} \right) = \frac{a^4}{3}$$

(b) Evaluate the same integral using polar coordinates. (Hint: make the substitution $u = \tan\theta$.)

in polar coordinates the function $x^2+y^2 = r^2$, the limits of integration are a little more involved, though:

$0 \le \theta \le \frac{\pi}{4}$

but r... depends on θ

when $x = a$, then since $x = r\cos\theta$ then the line $x = a$ corresponds to $r\cos\theta = a$,

or $r = \frac{a}{\cos\theta}$

so integral becomes $\int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} r^2\, r\,dr\,d\theta$

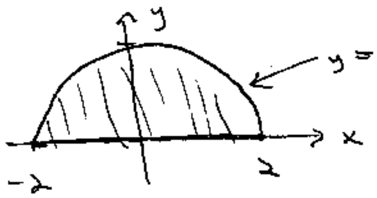
$$= \int_0^{\pi/4} \left(\frac{r^4}{4} \Big|_0^{\frac{a}{\cos\theta}} \right) d\theta = \frac{a^4}{4} \int_0^{\pi/4} \frac{1}{\cos^4\theta}\,d\theta$$ following suggestion

$u = \tan\theta$, $du = \frac{1}{\cos^2\theta}\,d\theta$ ... end up with $\frac{a^4}{4}\left(\frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x \right)\Big|_0^{\pi/4}$

$$= \frac{a^4}{4} \cdot \frac{4}{3} = \frac{a^4}{3}$$ (as before!)

(6) Convert the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dy\,dx$ to polar coordinates and hence evaluate it exactly. Sketch the region R over which the integration is being performed.

Sketch R first: $0 \le y \le \sqrt{4-x^2}$ as $-2 \le x \le 2$



$y=\sqrt{4-x^2}$ upper half of $x^2+y^2=4$, radius 2 circle

Then in polar coordinates we have $0 \le r \le 2$ with $0 \le \theta \le \pi$

and $e^{-(x^2+y^2)} = e^{-r^2}$

so integral becomes $\int_0^{\pi}\int_0^{2} e^{-r^2}\, r\,dr\,d\theta$

sub $u=-r^2$ $du=-2r\,dr$ $-\frac{1}{2}du = r\,dr$,

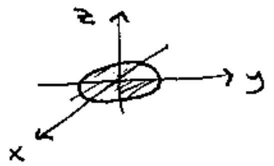
$\int r e^{-r^2}dr = \int -\frac{1}{2}e^u du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-r^2}$

so integral $= \int_0^{\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=2}\right)d\theta$

$= \int_0^{\pi}\left(-\frac{1}{2}e^{-4} - \left(-\frac{1}{2}\right)\right)d\theta = \frac{\pi}{2}\left(1-e^{-4}\right)$

(7) Suppose that a mass density function is given by $\delta(x,y,z) = x + z$. Set up, but do not evaluate, an iterated integral for the mass of the body which has this density function and which is bounded by the surfaces $x^2 + y^2 = 4$, $x + y + z = 5$, and $z = 1$.

From the $x^2+y^2=4$ equation we know the region "lives" over the radius 2 circle in the xy plane:



In terms of z we know that

$z = 5 - x - y$ (From $x+y+z=5$) is one of the bounds

and over the radius 2 circle (where x, y are at most equal to 2) that $5-x-y > 1$, so the lower bound for z is $z=1$, upper bound $5-x-y$, which depends on x, y, so do the z integral first in rectangular coordinates:

$$\iint_R \int_1^{5-x-y} (x+z)\, dz\, dA$$

($x+z$ ← $\delta(x,y,z)$)

where the rest of the integral is over the radius 2 circle $x^2+y^2 \le 4$

could do this in rectangular coordinates:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_1^{5-x-y} (x+z)\, dz\, dy\, dx$$

or convert everything to cylindrical, with $x = r\cos\theta$, $y = r\sin\theta$:

$$\int_0^{2\pi} \int_0^{2} \int_1^{5-r\cos\theta-r\sin\theta} (r\cos\theta + z)\, r\, dz\, dr\, d\theta$$