SOLUTIONS : FUNDAMENTAL THEOREM OF LINE INTEGRALS

SECTION ONE

(1) (a) Parametrize C by $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le \pi$.

Then $\int_C \underline{F} \cdot d\underline{r} = \int_{t=0}^{t=\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle = \underline{\underline{\pi}}$

(b) Parametrize C by $x = 1-t$, $y = 0$ $\quad 0 \le t \le 2$

Then $\int_C \underline{F} \cdot d\underline{r} = \int_0^2 \langle 0, 1-t \rangle \cdot \langle -1, 0 \rangle = \underline{\underline{0}}$

(2) We first find $f(x,y,z)$ such that $\underline{F} = \nabla f$ and then use the Fundamental Theorem of Line Integrals. We need ~~know that~~

$$\frac{\partial f}{\partial x} = yz \quad , \text{ so } \quad f(x,y,z) = xyz + g(y,z).$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y}$$

Comparing with the equation $\frac{\partial f}{\partial y} = xz$, we see that the function g depends only on z. Also

$$\frac{\partial f}{\partial z} = xy + \frac{dg}{dz}$$

so $\frac{dg}{dz} = 2z \quad \Rightarrow \quad g(z) = z^2 + C.$

Thus $f(x,y,z) = xyz + z^2 + C$. But now

$$\int_C \underline{F} \cdot d\underline{r} = \int_C (\nabla f) \cdot d\underline{r} = f(4,6,3) - f(1,0,-2)$$
$$= \underline{\underline{79}}$$

SECTION TWO

1 (a) Not conservative : $\frac{\partial P}{\partial y} = xe^y$ but $\frac{\partial Q}{\partial x} = ye^x$

(b) We solve $\frac{\partial f}{\partial x} = \sin xy + xy \cos xy$ --- (1)

$\frac{\partial f}{\partial y} = x^2 \cos xy$ ------ (2)

From (2), $f(x,y) = x \sin xy + g(x)$.

Differentiating this, we find

$$\frac{\partial f}{\partial x} = \sin xy + xy \cos xy + \frac{dg}{dx}$$

Comparing with (1) we see that $g(x)$ is a constant.

Thus $\underline{F} = \nabla (x \sin xy + C)$

SECTION THREE

(1) Easy

(2) Check that the integral $\int_{C_1} \underline{F} \cdot d\underline{r}$, where C_1 is the top half of the semicircle from $(1,0)$ to $(-1,0)$, is not equal to $\int_{C_2} \underline{F} \cdot d\underline{r}$, where C_2 is the bottom half of the same semicircle.

Thus $\int_C \underline{F} \cdot d\underline{r}$ is not independent of path, and so $\underline{F}$ is not conservative.

(3) $\underline{F}$ is not defined at the origin, and $\mathbb{R}^2$ minus the origin is not simply-connected.